Bayesian inference calibration of the modulus of elasticity

J. Dick[1] Q. T. Le Gia[2] K. Mustapha[3]

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Abstract

This work uses the Bayesian inference technique to infer the Young modulus from the stochastic linear elasticity equation. The Young modulus is modeled by a Karhunen–Loève expansion, while the solution to the linear elasticity equation is approximated by the finite element method. The high dimensional integral involving the posterior density and the quantity of interest is approximated by a higher-order quasi-Monte Carlo method.

Contents

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1 Introduction

The modulus of elasticity is a fundamental material parameter that characterizes stiffness and is critical in structural and mechanical design. However, its estimation is often subject to uncertainty due to measurement noise, material variability, and limited data availability. Traditional methods typically provide point estimates without accounting for these uncertainties, which can lead to overconfident predictions and potential design risks.

Bayesian inference [1, 2, 4, 8] offers a probabilistic framework for estimating the modulus of elasticity by combining observed data with prior knowledge. Instead of yielding a single value, it produces a full posterior distribution that captures the uncertainty in the estimate. This allows for more informed decision-making, particularly in applications involving heterogeneous materials, small sample sizes, or complex testing conditions. Moreover, Bayesian methods are inherently flexible, allowing for sequential updating as new data becomes available. This makes Bayesian inference a powerful and transparent tool for improving the reliability and interpretability of elasticity measurements in engineering and materials science.

The equation governing small elastic deformations of a polygonal body Ω in $\mathbb{R}^2$ with body force $\mathbf{f}$ and displacement vector field $\mathbf{u}(\cdot, \mathbf{y})$ satisfies

$$-\nabla \cdot \boldsymbol{\sigma}(\mathbf{y}; \mathbf{u}(\mathbf{x}, \mathbf{y})) = \mathbf{f}(\mathbf{x}), \quad \text{for } \mathbf{x} \in \Omega, \tag{1}$$

subject to homogeneous boundary conditions, $\mathbf{u}(\cdot, \mathbf{y}) = \mathbf{0}$ on $\partial\Omega$ (the boundary of Ω), where $\mathbf{y} = (y_j)_{j \geq 1}$ belongs to $U := (-\frac{1}{2}, \frac{1}{2})^{\mathbb{N}}$ consisting of a countable number of parameters y_j, which are assumed to be i.i.d. uniformly

distributed. The parametric Cauchy stress tensor $\boldsymbol{\sigma}(\boldsymbol{y};\cdot) \in [L^2(\Omega)]^{2\times 2}$ is defined as

$$\boldsymbol{\sigma}(\boldsymbol{y};\boldsymbol{u}(\cdot,\boldsymbol{y})) = \lambda(\cdot,\boldsymbol{y})\,[\nabla\cdot\boldsymbol{u}(\cdot,\boldsymbol{y})]\,I + 2\mu(\cdot,\boldsymbol{y})\boldsymbol{\varepsilon}(\boldsymbol{u}(\cdot,\boldsymbol{y})) \quad \text{on} \quad \Omega,$$

where I is the identity tensor and $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right]$ is the symmetric strain tensor. The gradient ∇ and the divergence $\nabla\cdot$ are understood to be with respect to the variable $\boldsymbol{x} \in \Omega$. The inhomogeneous random Lamé coefficients μ and λ are represented in terms of the Young's modulus E as

$$\mu(\boldsymbol{x},\boldsymbol{y}) = \frac{E(\boldsymbol{x},\boldsymbol{y})}{2(1+\nu)} \quad \text{and} \quad \lambda(\boldsymbol{x},\boldsymbol{y}) = \frac{E(\boldsymbol{x},\boldsymbol{y})\,\nu}{(1+\nu)(1-2\nu)}, \tag{2}$$

where the constant $0 < \nu < \frac{1}{2}$ is the Poisson ratio of the elastic material, and Young's modulus is defined by the Karhunen–Loève expansion

$$E(\boldsymbol{x},\boldsymbol{y}) = E_0(\boldsymbol{x}) + \sum_{j=1}^{\infty} y_j\psi_j(\boldsymbol{x}). \tag{3}$$

The ψ_j are $L^2(\Omega)$-orthogonal basis functions, which are assumed to belong to $L^\infty(\Omega)$.

In this work, we use the Bayesian inference technique to determine the conditional distribution of the solution to (1), given some data measurements. We consider the approximation of the conditional density with some quantity of interest applied to the solution using a higher-order quasi-Monte Carlo (QMC) method and finite element methods.

2 The forward problem

We introduce the following vector function spaces and the associated norms. Let $\boldsymbol{V} := [H_0^1(\Omega)]^2$, and the associated norm be $\|\cdot\|_{\boldsymbol{V}}$. For $J \geq 1$, the norm on the vector Sobolev space $\boldsymbol{H}^J := [H^J(\Omega)]^2$ is denoted by $\|\cdot\|_J$. For the $\boldsymbol{L}^2(\Omega)$-norm, we use the notation $\|\cdot\|$. Finally, $\boldsymbol{V}^*$ is denoted the dual

space of $\mathbf{V}$ with respect to the $\mathbf{L}^2(\Omega)$ inner product, with norm denoted by $\|\cdot\|_{\mathbf{V}^*}$.

The weak formulation for the forward problem is described as follows. Given $\mathbf{y} \in \mathbf{U}$, find $\mathbf{u}(\cdot, \mathbf{y}) \in \mathbf{V}$ satisfying

$$\mathcal{B}(\mathbf{y}; \mathbf{u}, \mathbf{v}) = \ell(\mathbf{v}), \quad \text{for all} \quad \mathbf{v} \in \mathbf{V}. \tag{4}$$

The bilinear form is defined by

$$\mathcal{B}(\mathbf{y}; \mathbf{u}, \mathbf{v}) := \int_\Omega [2\mu(\mathbf{y})\, \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) + \lambda(\mathbf{y}) \nabla \cdot \mathbf{u} \nabla \cdot \mathbf{v}]\, d\mathbf{x},$$

where the colon operator is the inner product between tensors, and the linear functional is defined by

$$\ell(\mathbf{v}) := \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x}.$$

For the well-posedness of the elastic problem (1), we assume throughout this article that $E_{\min} \le E \le E_{\max}$ on $\Omega \times U$, for some positive constants $E_{\min}$ and $E_{\max}$. This assumption leads to

$$\frac{E_{\min}}{2(1+\nu)} \le \mu \le \frac{E_{\max}}{2(1+\nu)} \quad \text{and} \quad \frac{\nu E_{\min}}{(1+\nu)(1-2\nu)} \le \lambda \le \frac{\nu E_{\max}}{(1+\nu)(1-2\nu)}$$

on $\Omega \times U$. Then, for every $f \in \mathbf{V}^*$ and $\mathbf{y} \in U$, the parametric weak formulation problem (4) has a unique solution [3, Theorem 2]. For a practical implementation, we truncate the Karhunen–Loève expansion of E in (3) by $E_s = E_0 + \sum_{j=1}^{s} y_j \psi_j$, that is, we assume that $y_j = 0$ for $j > s$. Then, with $\mathbf{y}_s = (y_1, y_2, \ldots, y_s, 0, 0, \ldots)$, and by using (2), the errors from truncating the Lamé coefficients are

$$\|\mu(\cdot, \mathbf{y}) - \mu(\cdot, \mathbf{y}_s)\|_{L^\infty(\Omega)} \le \frac{1}{2(1+\nu)} \sum_{j \ge s+1} \|\psi_j\|_{L^\infty(\Omega)} \quad \text{and}$$

$$\|\lambda(\cdot, \mathbf{y}) - \lambda(\cdot, \mathbf{y}_s)\|_{L^\infty(\Omega)} \le \frac{1}{1-2\nu} \sum_{j \ge s+1} \|\psi_j\|_{L^\infty(\Omega)}.$$

For the convergence of the series truncation in (3), we order the functions $\|\psi_j\|_{L^\infty(\Omega)}$ in a decreasing order, that is, $\|\psi_j\|_{L^\infty(\Omega)} \geq \|\psi_{j+1}\|_{L^\infty(\Omega)}$ for $j \geq 1$. In addition, we assume that for some $0 < p < 1$:

$$\sum_{j=1}^{\infty} \|\psi_j\|_{L^\infty(\Omega)}^p < \infty \quad \text{or} \quad \sum_{j=s+1}^{\infty} \|\psi_j\|_{L^\infty(\Omega)} \leq C s^{1-1/p}. \tag{5}$$

For the Galerkin finite element convergence analysis, assume that

$$E_0 \in W^{J,\infty}(\Omega), \quad \sum_{j=1}^{\infty} \|\psi_j\|_{W^{J,\infty}(\Omega)} < \infty. \tag{6}$$

The weak formulation for the truncated problem is: for every $\mathbf{y}_s \in U$, find $\tilde{\mathbf{u}}(\cdot, \mathbf{y}_s) \in V$ satisfying

$$\mathcal{B}(\mathbf{y}_s; \tilde{\mathbf{u}}, \mathbf{v}) = \ell(\mathbf{v}), \quad \text{for all} \quad \mathbf{v} \in V. \tag{7}$$

Following the proof of Clarke et al. [3, Theorem 5], we obtain error results in the following theorem. The constant C depends on Ω, ν, $E_{\min}$ and $E_{\max}$.

Theorem 1. *Assume that (5) is satisfied for some $0 < p < 1$, and $\chi : V \to \mathbb{R}$ is a bounded linear functional, $|\chi(\mathbf{w})| \leq \|\chi\|_{V^*}\|\mathbf{w}\|_V$ for all $\mathbf{w} \in V$. Then, for every $f \in V^*$, $\mathbf{y} \in U$, and $s \in \mathbb{N}$, the solution $\tilde{\mathbf{u}}$ of the truncated parametric weak problem (7) satisfies*

$$|\chi(\mathbf{u}(\cdot, \mathbf{y})) - \chi(\tilde{\mathbf{u}}(\cdot, \mathbf{y}_s))| \leq C\, s^{1-1/p}\, \|f\|_{V^*}\|\chi\|_{V^*}.$$

Next, we approximate the truncated solution $\tilde{\mathbf{u}}$ using the Galerkin finite element solution of degree at most r with $r \geq 1$. So, we introduce a family of regular triangulation (made of simplexes) $\mathcal{T}_h$ of the domain $\bar{\Omega}$ and set $h = \max_{\rho \in \mathcal{T}_h}(h_\rho)$, where h_ρ denotes the diameter of the mesh element ρ. Let $V_h \subset H_0^1(\Omega)$ denote the usual conforming finite element space of continuous, piecewise polynomial functions of degree at most r on $\mathcal{T}_h$ that vanish

on $\partial\Omega$. Let $\mathbf{V}_h = [V_h]^d$ be the finite element vector space. We define the parametric finite element approximate solution as: find $\tilde{\mathbf{u}}_h(\cdot, \mathbf{y}_s) \in \mathbf{V}_h$ such that

$$\mathcal{B}(\mathbf{y}_s; \tilde{\mathbf{u}}_h, \boldsymbol{v}_h) = \ell(\boldsymbol{v}_h), \quad \text{for all} \quad \boldsymbol{v}_h \in \mathbf{V}_h, \quad \text{for every } \mathbf{y}_s \in U. \tag{8}$$

In the following theorem, we discuss the error estimate from the finite element discretization. For the proof, we refer to Clarke et al. [3, Theorem 6]. We assume that the truncated continuous solution $\tilde{\mathbf{u}}$ satisfies the regularity property: for some $1 \le J \le r$,

$$\|\tilde{\mathbf{u}}(\cdot, \mathbf{y}_s)\|_{J+1} \le C\|\mathbf{f}\|_{J-1}, \quad \text{for every } \mathbf{y}_s \in U, \tag{9}$$

where $\|\cdot\|_0 = \|\cdot\|$.

Theorem 2. *Assume that (6) and (9) are satisfied. If $\chi : \mathbf{L}^2(\Omega) \to \mathbb{R}$ is a bounded linear functional, then for every $\mathbf{y}_s \in U$, we have*

$$|\chi(\tilde{\mathbf{u}}(\cdot, \mathbf{y}_s)) - \chi(\tilde{\mathbf{u}}_h(\cdot, \mathbf{y}_s))| \le Ch^{J+1}\|\mathbf{f}\|_{J-1}\|\chi\|, \quad \textit{for } 1 \le J \le r.$$

The constant C depends on Ω, ν, $E_{\max}$ and $E_{\min}$, but not on h.

So far, we have discussed the error from truncating the infinite series expansion in (2) and the error from approximating the truncated solution via the finite element method. The next task is to estimate the expected value of $\chi(\tilde{\mathbf{u}}_h)$ using a higher-order QMC rule.

3 Higher-order QMC method

We use an N-point QMC rule $Q_{N,s}(F)$ to approximate an s-dimensional integral:

$$I_s(F) := \int_{[0,1]^s} F(\mathbf{y})\, d\mathbf{y} \approx Q_{N,s}(F) := \frac{1}{N}\sum_{n=0}^{N-1} F(\mathbf{y}_n). \tag{10}$$

The QMC points $\{\mathbf{y}_0, \ldots, \mathbf{y}_{N-1}\}$ belong to $[0,1)^s$. We analyze, in particular, deterministic $Q_{N,s}$ using *interlaced high-order polynomial lattice* rules, as introduced by Dick [5] and as considered for affine-parametric operator equations by Dick et al. [7].

Before we introduce the polynomial lattice rule, we need some notation. Let $\mathbb{N}_0$ be the set of all non-negative integers. Let b be prime and $\mathbb{Z}_b$ be the finite field with b elements. For convenience, we identify the elements in $\mathbb{Z}_b$ with the integers $\{0, 1, \ldots, b-1\}$. Let $\mathbb{Z}_b[x] = \left\{ \sum_{i=0}^{k} a_i x^i \mid k \in \mathbb{N}_0, a_i \in \mathbb{Z}_b \right\}$, be the set of all polynomials with coefficients in $\mathbb{Z}_b$, and let $G_{b,m}$ be the set of polynomials in $\mathbb{Z}_b[x]$ of degree less than m. For each integer n such that $0 \le n \le b^m - 1$, we associate the polynomial $n(x) = \sum_{i=0}^{m-1} \eta_i x^i \in G_{b,m}$ and have the b-adic expansion $n = \sum_{i=0}^{m-1} \eta_i b^i$.

For polynomials $p, q \in \mathbb{Z}_b[x]$, we consider the Laurent series expansion of the rational function

$$\frac{q(x)}{p(x)} = \sum_{\ell=w}^{\infty} t_\ell x^{-\ell} \in \mathbb{Z}_b((x^{-1})) \quad \text{for some integer } w,$$

where $\mathbb{Z}_b((x^{-1}))$ is the field of formal Laurent series over $\mathbb{Z}_b$.

We also need ν_m which maps elements in $\mathbb{Z}_b((x^{-1}))$ to the interval $[0,1)$. This map is defined by $\nu_m\left(\sum_{\ell=w}^{\infty} t_\ell x^{-\ell}\right) = \sum_{\ell=\max(1,w)}^{m} t_\ell b^{-\ell}$, where for any integer $w > m$ we set $\nu_m\left(\sum_{\ell=w}^{\infty} t_\ell x^{-\ell}\right) = 0$.

Let $m \in \mathbb{N}$. To generate a polynomial lattice rule in prime base b with $N = b^m$ points in $[0,1)^s$, we need a *generating vector* of polynomials $g(x) = (g_1(x), \ldots, g_s(x))$ with $g_i \in G_{b,m}$ for $1 \le i \le s$, and an irreducible polynomial $P \in \mathbb{Z}_b[x]$ with degree m. The classical polynomial lattice rule $\mathcal{S}_{P,b,m,s}(g)$ associated with P and the generating vector g is comprised of the quadrature points

$$y_n = \left[\nu_m\left(\frac{n(x) g_j(x)}{P(x)} \right) \right]_{1 \le j \le s} \in [0,1)^s, \quad \text{for } n = 0, \ldots, N-1.$$

Classical polynomial lattice rules give almost first order convergence for integrands of bounded variation. To obtain high-order convergence, an interlacing procedure is needed. Following Goda and Dick [10], our interlacing procedure uses the *digit interlacing function* $\mathcal{D}_\alpha : [0,1)^\alpha \to [0,1)$ with digit interlacing factor $\alpha \in \mathbb{N}$ is defined by

$$\mathcal{D}_\alpha(x_1, \dots, x_\alpha) = \sum_{i=1}^{\infty} \sum_{j=1}^{\alpha} \xi_{j,i} b^{-j-(i-1)\alpha},$$

where $x_j = \sum_{i\geq 1} \xi_{j,i} b^{-i}$ for $1 \leq j \leq \alpha$. For vectors, we set $\mathcal{D}_\alpha^s : [0,1)^{\alpha s} \to [0,1)^s$ with $\mathcal{D}_\alpha^s(x_1, \dots, x_{\alpha s}) = (\mathcal{D}_\alpha(x_1, \dots, x_\alpha), \dots, \mathcal{D}_\alpha(x_{(s-1)\alpha+1}, \dots, x_{s\alpha}))$. Then, an interlaced polynomial lattice rule of order α with b^m points in s dimensions is a QMC rule using $\mathcal{D}_\alpha(\mathcal{S}_{P,b,m,\alpha s}(g))$ as quadrature points, for some given modulus P and generating vector g.

The following theorem states an error bound for approximating the integral $\mathcal{I}_s(F)$ by the QMC quadrature formula $Q_{s,N}(F)$ in (10). The proof relies on Theorem 3.1 by Dick et al. [7].

Theorem 3. *Let $\chi = (\chi_j)_{j\geq 1}$ be a sequence of positive numbers such that $\chi_j \geq \chi_{j+1}$ for all $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} \chi_j^p$ being finite for some $0 < p < 1$. Let $\chi_s = (\chi_j)_{1\leq j\leq s}$ and let $\alpha := \lfloor 1/p \rfloor + 1$. Assume that F satisfies the regularity property: for any $y \in [0,1)^s$ and any $\boldsymbol{\alpha} \in \{0, 1, \dots, \alpha\}^s$, $|\partial_y^{\boldsymbol{\alpha}} F(y)| \leq c|\boldsymbol{\alpha}|! \chi_s^{\boldsymbol{\alpha}}$, where the constant c is independent of y, s and p. Then one can construct an interlaced polynomial lattice rule of order α with N points, using a fast component-by-component algorithm, with cost $\mathcal{O}(\alpha s N(\log N + \alpha s))$ operations, so that*

$$|\mathcal{I}_s(F) - Q_{s,N}(F)| \leq C N^{-1/p}.$$

The constant C depends on b and p, but is independent of s and m.

4 Bayesian inversion

We consider the following *inverse problem*: given observational data δ, predict a 'most likely' value of a Quantity of Interest (QoI) ϕ, which is typically a continuously differentiable functional of the uncertain Young's modulus E. Let $\mathcal{G}$ be the solution operator of equation (1). That is $\mathcal{G}(E) = \mathbf{u}$.

We are given an observation functional $\mathcal{O}(\,\cdot\,) : \mathbf{V} \to Y$ (these are the bounded linear functionals χ in the forward problem), which denotes a *bounded linear observation operator* on the space $\mathbf{V}$ of observed system responses in Y. We assume that there is a finite number K of sensors, and each sensor measures the values of two components of the data, so that $Y = \mathbb{R}^{2K}$. Then $\mathcal{O} \in \mathcal{L}(\mathbf{V}; Y) \simeq (\mathbf{V}^*)^{2K}$. We equip Y with the Euclidean norm, denoted by $\|\cdot\|$. For example, if $\mathcal{O}(\,\cdot\,)$ is a K-vector of observation functionals then $\mathcal{O}(\,\cdot\,) = (o_k(\,\cdot\,))_{k=1}^{2K}$.

We want to solve the inverse problem of finding E from $\delta = \mathcal{O}(\mathcal{G}(E)) + \eta$, where E has probability measure π_0 on U (so $E \sim \pi_0$ on U), and additive Gaussian noise $\eta \sim \mathcal{N}(0, \Gamma)$ has a multivariate normal distribution on $\mathbb{R}^{2K}$ with covariance matrix Γ. The covariance-weighted, least squares potential $\Phi_\Gamma : \mathbf{V} \times Y \to \mathbb{R}$ is then

$$\Phi_\Gamma(E; \delta) = \frac{1}{2}\|\delta - \mathcal{O}(\mathcal{G}(E))\|_\Gamma^2 := \frac{1}{2}\left[(\delta - \mathcal{O}(\mathcal{G}(E)))^\top \Gamma^{-1}(\delta - \mathcal{O}(\mathcal{G}(E)))\right].$$

The following theorem is a variant of Theorem 14 by Dashti and Stuart [4]

Theorem 4. *Assume that the potential* $\Phi_\Gamma(E; \delta) = \frac{1}{2}\|\delta - \mathcal{O}(\mathcal{G}(E))\|_\Gamma^2$ *is* $\boldsymbol{\pi}_0$ *measurable and that for* δ, $\mathcal{N}(0, \Gamma)$*-almost surely, it holds that*

$$Z := \int_U \exp\left(-\Phi_\Gamma(E; \delta)\right) \boldsymbol{\pi}_0(du) > 0.$$

Then the conditional distribution of $\mathbf{u} \mid \delta$ *(or* $\mathbf{u}$ *given* δ*) exists and is denoted by* $\boldsymbol{\pi}^\delta$*. This distribution is continuous with respect to* $\boldsymbol{\pi}_0$ *and*

$$\frac{d\boldsymbol{\pi}^\delta}{d\boldsymbol{\pi}_0}(E) = \frac{1}{Z}\exp\left(-\Phi_\Gamma(E; \delta)\right).$$

With the QoI $\phi : \mathbf{L}^2(\Omega) \to \mathbb{R}$, we associate the parametric map

$$\Psi(\mathbf{y}) = \Theta(\mathbf{y})\phi(\mathbf{u}(\mathsf{E})) = \exp\Big(-\Phi_\Gamma(\mathsf{E};\delta)\Big)\phi(\mathbf{u}(\mathsf{E}))\,,$$

where

$$\Theta(\mathbf{y}) = \exp\left(-\Phi_\Gamma(\mathsf{E};\delta)\right).$$

The Bayesian estimate of the QoI ϕ, given noisy data δ, takes the form

$$\mathbb{E}^{\pi^\delta}[\phi] = \mathsf{Z}'/\mathsf{Z}\,, \quad \mathsf{Z}' := \mathrm{I}(\Psi) = \int_{\mathrm{U}} \Psi(\mathbf{y})\,\boldsymbol{\pi}_0(\mathrm{d}\mathbf{y})\,.$$

We now propose an approximation strategy to compute the high dimensional integrals Z and Z' using a higher-order QMC method. The truncated parametric map applied to the truncated forward problem is defined by

$$\Psi(\mathbf{y}_s) = \Theta(\mathbf{y}_s)\phi(\tilde{\mathbf{u}}(\mathsf{E}_s))\,, \quad \text{with } \Theta(\mathbf{y}_s) = \exp\left(-\Phi_\Gamma(\mathsf{E}_s;\delta)\right).$$

Thus, the truncated parametric map applied to the finite element approximation to the forward problem is defined by

$$\Psi_h(\mathbf{y}_s) = \Theta(\mathbf{y}_s)\phi\left(\tilde{\mathbf{u}}_h(\mathsf{E}_s)\right).$$

Theorem 5. *Assume that (5), (6) and (9) are satisfied. In addition, the QoI mapping ϕ is assumed to be a bounded linear functional. Let $\mathsf{Z}'_{N,s,h} = \mathrm{Q}_{N,s}(\Psi_h)$, and $\mathsf{Z}' = \mathrm{I}(\Psi)$, then we have the error bound*

$$|\mathsf{Z}'_{N,s,h} - \mathsf{Z}'| \le C(h^{J+1} + N^{-1/p} + s^{1-1/p})\,, \quad \textit{for } 1 \le J \le r\,.$$

Proof: Let $\mathsf{Z}'_{N,s} = \mathrm{Q}_{N,s}(\Psi)$ and $\mathsf{Z}'_s = \mathrm{I}_s(\Psi)$. Then, by using the triangle inequality, we have

$$|\mathsf{Z}'_{N,s,h} - \mathsf{Z}'| \le |\mathsf{Z}'_{N,s,h} - \mathsf{Z}'_{N,s}| + |\mathsf{Z}'_{N,s} - \mathsf{Z}'_s| + |\mathsf{Z}'_s - \mathsf{Z}'|\,.$$

We now estimate term by term. An application of Theorem 2 yields

$$|\mathsf{Z}'_{N,s,h} - \mathsf{Z}'_{N,s}| = |\mathrm{Q}_{N,s}(\Psi_h) - \mathrm{Q}_{N,s}(\Psi)| \le \|\Psi_h - \Psi\|_{\mathrm{L}^\infty}$$

$$\leq \|\Theta\|_{L^\infty} |\phi(\tilde{\mathbf{u}}) - \phi(\tilde{\mathbf{u}}_h)| \leq C h^{J+1} \|\Theta\|_{L^\infty} \|\phi\| \|\mathbf{f}\|_{J-1} .$$

To estimate the quadrature error, we use Theorem 3 and obtain

$$|Z'_{N,s} - Z'_s| = |Q_{N,s}(\Psi) - I_s(\Psi)| \leq C N^{-1/p} .$$

Using Theorem 1 the truncation error is estimated as

$$|Z'_s - Z'| = |I_s(\Psi) - I(\Psi)| \leq C \max_{\mathbf{y} \in U} |\Theta(\mathbf{y})| |\phi(\mathbf{u}) - \phi(\tilde{\mathbf{u}})| \leq C s^{1-1/p} .$$

Combining all error estimates, we arrive at the result of the theorem. ♠

In Theorem 5, we assume that the normalizing constant Z is known exactly, or at least can be approximated sufficiently accurately so as not to influence the error. Dick et al. [6] discuss an error analysis in which the error arising from approximating Z is also included.

5 Numerical experiments

We choose unit square $\Omega = [0,1]^2$. Assume that the Young's modulus depends on the parameters $\mathbf{y} = \{y_j\}_{j=1}^s$ via the finite expansion (so $\mathbf{u} = \tilde{\mathbf{u}}$)

$$E(\mathbf{x}, \mathbf{y}) = 1 + \sum_{j=1}^{s} \frac{y_j}{j^2} \sin(2\pi j x_1) \sin(2\pi(j+1)x_2) , \quad \mathbf{x} = (x_1, x_2) \in \Omega .$$

As prior measure π_0 on $(-1/2, 1/2)^s$, we use the uniform product measure $\pi_0(du) = \otimes_{j=1}^s dy_j$.

We set the Poisson ratio $\nu = 0.4$ and $\mathbf{f}(\mathbf{x}) = (2x_1 + 10,\ x_2 - 3)$. The forward map $\mathcal{G} : W^{J,\infty}(\Omega) \to \mathbf{V}$, defined by $\mathcal{G}(E) = \mathbf{u}$, is approximated by the quadratic finite element solver $\mathcal{G}_h(E) = \tilde{\mathbf{u}}_h$. We choose $K = 10$ observation points with Cartesian coordinates $\mathbf{x}_k = (0.5, 10^{-3} + k(10^{-1} - 10^{-4}))$ for $k = 0, 1, \ldots, 9$. The noisy observations are point evaluations of the solution

$$o_k = \mathbf{u}(\mathbf{x}_k, \mathbf{y}^*) + \eta_k , \quad k = 0, \ldots, 9.$$

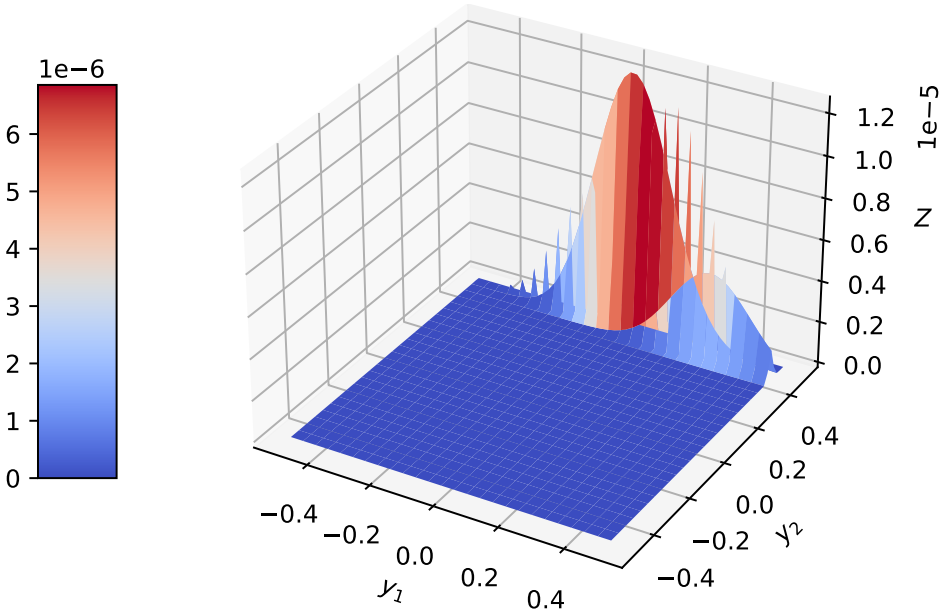

Figure 1: Un-normalised posterior density with $s = 2$.

with $\eta_k \sim \mathcal{N}(0, \Gamma)$. In the experiments, we choose $\Gamma = \sigma \mathbf{I}$, with $\sigma = 0.1$ and $\mathbf{I}$ the $(2K) \times (2K)$ identity matrix. To generate the observations, the truth $\mathbf{y}^*$ was chosen as a sample in the prior. In Figure 1, we set $s = 2$ and plot the un-normalized posterior density, that is, the function $\exp(-\Phi_\Gamma(E; \delta))$ over the uniform grid $(-1/2, 1/2)^2$. With $\mathbf{u} = (u_1, u_2)^\top$ we define the linear functional

$$\phi(\mathbf{u}) = \int_\Omega (u_1 + u_2) d\mathbf{x}.$$

We then set $s = 64$, generate another set of measurements with noise and compute the approximate integrals Z'_N/Z with different values of N. We use higher-order QMC point sets based on those used by Gantner and Schwab [9]. Table 1 shows the errors of Z'_N/Z compared to the reference value (computed with $N = 2^{15} = 32\,768$ quadrature points) with numerical convergence order.

Table 1: Errors of Z'_N/Z compared to the reference value.

N	error	Convergence order
256	0.8758	
1024	0.3500	0.6617
2048	0.1193	1.5527
4096	0.0176	2.7612

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Author addresses

1. **J. Dick**, School of Mathematics and Statistics, UNSW, Sydney, Australia AUSTRALIA.
mailto:josef.dick@unsw.edu.au

2. **Q. T. Le Gia**, School of Mathematics and Statistics, UNSW, Sydney, Australia.
mailto:qlegia@unsw.edu.au

3. **K. Mustapha**, School of Mathematics and Statistics, UNSW, Sydney, Australia.
mailto:kassem.ahmad.mustapha@gmail.com