Solving for railway deformations using Weeks' method and undetermined coefficients

J. Murray[1] R. Pethiyagoda[2] M. H. Meylan[3]
N. Thamwattana[4]

(Received 19 February 2025; revised 28 August 2025)

Abstract

With the increasing demands being placed on railway infrastructure, we are motivated to provide readily accessible solutions to models which can be used to describe the mechanics of rail track, particularly at transition zones. Transition zones, where foundation materials change abruptly along the track, are of particular interest as they have to be maintained as much as eight times more frequently than standard track sections. Modelling the railway as an infinite Euler–Bernoulli beam on a viscoelastic foundation, we apply the Laplace transform to eliminate time derivatives and solve the resultant ordinary differential equation in space by the method of undetermined coefficients. The Laplace transform must then be inverted by a numerical technique, to which end we provide a practical description of Weeks' method. The

DOI:10.21914/anziamproc.v66.19623, © Austral. Mathematical Soc. 2025. Published 2025-12-08, as part of the Proceedings of the 22nd Biennial Computational Techniques and Applications Conference. ISSN 1445-8810. (Print two pages per sheet of paper.) Copies of this article must not be made otherwise available on the internet; instead link directly to the DOI for this article.

results obtained through the application of this method are compared to published solutions for the steady-state deformation of rail track at transition zones.

Contents

1 Introduction

As populations increase, the need for efficient rail systems increases in turn [8]. In response, high-speed rail networks—where speeds exceed 200 km/h—are becoming increasingly common throughout Europe and Asia. Notably, these networks do not exist in Australia, where the current infrastructure is inadequate to support such speeds, limiting trains to a maximum of 160 km/h [9]. Heavy haul networks, another response, are also becoming more common globally, but in this area Australia is playing a leading role [11]. In both cases there is increased mechanical demand on rail systems and especially on sleeper and ballast infrastructure, which is widely used in Australia across its over 30 000 km long rail network [12]. This increased demand, in turn, necessitates either increased maintenance, or improved rail infrastructure. As such, improving rail infrastructure is an active area of research in Australia and globally.

Particular attention may be paid to places along the tracks where the foundation changes abruptly—known as transition zones. Common examples of transition zones include the ends of bridges, culverts, and level crossings, where the track may change from being supported by deep ballast on soft soils to being supported by much shallower ballast or directly by concrete slabs. The differing foundational properties mean that plastic deformations accumulate at different rates across the transition zone, creating differential settlements which increase dynamic loads, track deterioration, and passenger discomfort. This necessitates more frequent maintenance [14, 17]. In fact, transition zones may have to be maintained as much as eight times as frequently as standard track sections [10]. Research into approaches for mitigating the differential settlement across transition zones is ongoing and motivates a better understanding of the mechanics of the system [17, 14, 21].

Straight sections of railway infrastructure may be conveniently modelled using one dimensional beam models [7]. A common choice is an Euler–Bernoulli beam on a viscoelastic foundation, however, readily available solutions to this model are somewhat scarce. In this article, we present a preliminary method for simple problems which may be easily extended. Specifically, we propose the use of the Laplace transform and numerical inversion methods to solve the time-dependency of the governing differential equation together with the method of undetermined coefficients to solve the equation in space. The Laplace transform has been chosen ahead of purely numerical methods, as even when it must be inverted numerically—as in this article—it still avoids errors associated with time-stepping. For problems such as the long-term plastic settlement at transition zones, deformations occurring on second-long time-scales can impact the behaviour of the system on the order of years [18]. Make the time-step too large and one risks losing the impact of small time-scales. Make the time-step too small and the computation times become impractically large. In contrast to this, the numerical inversion of the Laplace transform can be found with relative accuracy for arbitrary timescales, although asymptotic methods are suggested for especially large or small values [1, 5].

The major drawback of the numerical Laplace transform is its inherent difficulty. In general, the problem of finding the inverse Laplace transform is ill-conditioned [13] and, indeed, it still rings true that '…Laplace transform inversion is still more of an art than a science' [4]. As a result, it is often recommended that more than one numerical inversion method is used where a known solution is not available [3, 4], and as such there is a great number which exist in the literature. For the choice of numerical Laplace inversion method, Gaver–Stehfest is a common choice amongst articles concerning Laplace-space problems, however Khulman's review [13] recommends other methods, depending on the researchers priorities. Khulman recommends the Fourier series method for good convergence for a variety of behaviours, Talbot method for ease of implementation, and Weeks' method if the solution values are required for a large number of time values. This last point is maintained throughout the literature [3, 5], and, as we consider it to be important here, Weeks' method is chosen for implementation in this study.

Weeks' method [20] seeks to invert the Laplace transform by expanding the target solution in terms of Laguerre polynomials. Unlike many other methods—such as Talbot and Gaver–Stehfest—this results in an approximating function, rather than a solution value at a particular time. Hence, to find the solution for other time values, the approximating function can be reevaluated, rather than having to repeat the inversion process. In all cases, implementation of Weeks' method amounts to calculating the coefficients of the series of Laguerre polynomials, however, the details of how the method is derived and implemented varies somewhat in the literature. The most significant difference is in Piessen and Branders' work [16], where the method is extended to use generalised Laguerre polynomials. Other variations only constitute a change to the particular quadrature method used in the evaluation of the coefficients of the approximating series [15]. In this article, we use a modification [6, 15] which roughly halves the number of evaluations which have to be performed on the Laplace domain function.

The purpose of this article is to provide a ready summary of some practical points for the implementation of Weeks' method, and demonstrate its use in solving for the deformation of an Euler–Bernoulli beam on a viscoelastic foundation and subject to a stationary point load. An overview of Weeks' method and the modification by Lyness and Giunta [15] is provided in Section 2, which also contains a subsection on the necessary Clenshaw algorithm (Section 2.1). A description of the simple beam problem considered here is given in Section 3. Results for the application of Weeks' method to this problem are given in Section 4 before some concluding remarks in Section 5.

2 Weeks' method

Weeks' method begins by supposing that the unknown time domain solution can be expressed in terms of Laguerre functions in the form

$$f(t) = e^{\sigma t} \sum_{k=0}^{\infty} a_k e^{-bt/2} L_k(bt) \,, \tag{1}$$

where σ and b are real valued control parameters which need to be chosen, $L_k(t)$ is the kth Laguerre polynomial, and a_k are coefficients. The method involves selecting appropriate b and σ, calculating the unknown coefficients a_k, and computing the solution using equation (1). For this article, we choose $\sigma = (\sigma_0 + 1/t_{\max})H(\sigma_0 + 1/t_{\max})$, where $t_{\max}$ is the largest time up to which we solve, $H(\,\cdot\,)$ is the Heaviside step function, and σ_0 is the real component of the rightmost pole of the Laplace domain function $\bar{f}(s)$. This is as recommended by Weeks [20], noting that Weeks' variables c and c_0 are equivalent to the parameters σ and σ_0, used here. We also use $b = 2(\sigma - \sigma_0)$, as suggested by Garbow et al. [6]. We focus here on presenting a derivation of the process for finding the a_k values, as it is presented by Lyness and Giunta [15], which we believe to be the clearest formulation of the method.

In particular, we apply the Laplace transform to equation (1) and truncate the series to N terms, arriving at a series approximation for our known Laplace domain function

$$\bar{f}(s) \approx \sum_{k=0}^{N} a_k \frac{(s-\sigma-b/2)^k}{(s-\sigma+b/2)^{k+1}}\,, \tag{2}$$

which involves the unknown coefficients a_k. In this form, we note that the right-hand side of (2) is nearly a series involving powers of a Möbius transform in s.

By applying the substitution

$$z = \frac{(s-\sigma-b/2)}{(s-\sigma+b/2)}\,, \quad \text{or equivalently} \quad s = \frac{b}{1-z} - \frac{b}{2} + \sigma\,, \tag{3}$$

we arrive at the expression

$$\phi(z) = \frac{b}{1-z}\bar{f}\left(\frac{b}{1-z} - \frac{b}{2} + \sigma\right) = \sum_{k=0}^{\infty} a_k z^k\,,$$

which is a power series centred at $z = 0$ for a known function. This is especially convenient as the transformation (3) maps the half plane $\operatorname{Re}(s) > \sigma$ to the unit disk. When the parameter σ is chosen so that it is to the right of the right-most pole, $\phi(z)$ is analytic on the unit disk. As such, we can apply the Cauchy integral form of the derivative to calculate the coefficients

$$a_k = \frac{\phi^{(k)}(0)}{k!} = \frac{1}{2\pi i}\int_C \frac{\phi(z)}{z^{k+1}}\,dz\,,$$

using a contour integral on the unit circle.

Different quadrature schemes can be used to evaluate the contour, but we use here a modification which roughly halves the number of function evaluations which need to be performed [15]. In particular, Lyness and Giunta [15] note that

$$\overline{\phi(z)} = \phi(\bar{z})\,.$$

This allows for the approximation

$$a_k \approx \frac{1}{r^k N}\Big(\phi^*(r) + (-1)^k\phi(-r) + 2\sum_{j=1}^{N/2-1} \Re[\phi(re^{2\pi ij/N})]\cos(2\pi kj/N) \\ + \Im[\phi(re^{2\pi ij/N})]\sin(2\pi kj/N)\Big), \tag{4}$$

which is an N point quadrature rule that requires only $N/2+1$ evaluations of the complex function $\phi(z)$. Notably, from this derivation, only equation (4) is necessary for the computation of the coefficients a_k.

2.1 Clenshaw algorithm

While direct computation of the Weeks approximation (1) is possible, sums of Laguerre polynomials are notoriously prone to cancellation error, and as such it is recommended that the Clenshaw algorithm be implemented [3]. The Clenshaw algorithm [2], also referred to as the Clenshaw recurrence formula (e.g., in Khulman's work [13]), is a convenient method to evaluate series of recursive functions. Whilst originally published for use with Chebychev polynomials, it can be applied to any functions with a recursion formula of the type

$$p_{n+1}(x) + \alpha_n(x)p_n(x) + \beta_n(x)p_{n-1}(x) = 0\,,$$

where p_j are the recursively defined functions.

Say we have a series

$$f_N(x) = \sum_{n=0}^{N} a_n p_n(x)\,,$$

then for the Clenshaw algorithm we construct a back-propagating sequence $b_N, b_N - 1, \ldots, b_0$, adding $b_{N+1} = b_{N+2} = 0$, satisfying

$$b_n(x) = a_n - \alpha_n(x)b_{n+1}(x) - \beta_{n+1}(x)b_{n+2}(x)\,,$$

and then express the series as

$$f_N(x) = b_0(x)p_0(x) + b_1(x)[p_1(z) + \alpha_0(x)p_0(x)] .$$

One advantage of this method is that it allows the evaluation of the series without needing to calculate the entire sequence of functions, $p_n(x)$. In the context of this article, we are interested in Laguerre polynomials which have

$$\alpha_n(x) = \frac{1}{n+1}x - \frac{2n+1}{n+1} ,$$
$$\beta_n(x) = \frac{-n}{n+1} ,$$

and $p_0(x) = 1$, $p_1(x) = 1 - x$. The Clenshaw algorithm then involves the calculation of the sequence

$$b_n(x) = a_n - \left(\frac{1}{n+1}x - \frac{2n+1}{n+1}\right) b_{n+1}(x) + \frac{n}{n+1} b_{n+2}(x) ,$$

and has solution

$$f_N(x) = b_0(x) .$$

3 Point force on an infinite beam

The governing equation for the time-dependent deformation of an Euler–Bernoulli beam resting on a viscoelastic foundation is given by

$$EI\partial_x^4 y + m\partial_t^2 y + C(x)\partial_t y + k(x)y = F(x,t) , \tag{5}$$

where x is the position along the axis of the beam, t is time, y is the vertical beam deformation, E is the Young's modulus of the beam, I is the moment of inertia of the beam, m is the mass per unit length of the beam, C is a damping coefficient, $k(x)$ is the stiffness of the foundation, and $F(x,t)$ is an applied load. In the interest of providing a simple case to which we can demonstrate an application of Weeks' method and undetermined coefficients, we take $C(x) = C$ and $k(x) = k$ to be constant and consider the case of a single stationary point force at the origin, in which we take the force to be modelled by a dirac delta function,

$$F(x,t) = P\delta(x)\,.$$

We also impose zero deformation in the limit as x goes to infinity, with the beam initially at rest, and we suppose the deformation to be twice continuously differentiable, though this is only important at the point force. In summary, we consider the problem

$$EI\partial_x^4 y + m\partial_t^2 y + C\partial_t y + ky = P\delta(x)\,, \tag{6a}$$
$$y(x,0) = 0\,, \quad \forall x \in \mathbb{R}\,, \tag{6b}$$
$$\partial_t y(x,t)|_{t=0} = 0\,, \quad \forall x \in \mathbb{R}\,, \tag{6c}$$
$$\lim_{|x|\to\infty} y(x,t) = 0\,, \quad \forall t \in [0,\infty)\,, \tag{6d}$$
$$y(0^+,t) = y(0^-,t)\,, \quad \forall t \in [0,\infty)\,, \tag{6e}$$
$$\partial_t^j y(x,t)|_{x=0^+} = \partial_t^j y(,t)|_{x=0^-}\,, \quad j = 1,2,\ \forall t \in [0,\infty)\,. \tag{6f}$$

Applying the Laplace transform to (6a), we obtain

$$EI\partial_x^4 \bar{y} + m[s^2\bar{y} - sy(x,0) - \partial_t y|_{t=0}] + C[s - y(x,0)] + k\bar{y} = \tfrac{P}{s}\delta(x)\,,$$

which simplifies with the initial conditions, (6b) and (6c), to

$$EI\partial_x^4 \bar{y} + (ms^2 + Cs + k)\bar{y} = \tfrac{P}{s}\delta(x)\,. \tag{7}$$

Equation (7) is then an ordinary differential equation in x, to which we can apply the method of undetermined coefficients, before inverting the Laplace transform. To apply the method of undetermined coefficients, we first divide the solution domain into two regions, split by the point force. This leads to the general solution

$$\bar{y}(x,s) = \begin{cases} \sum_{j=1}^4 b_j(s)e^{r_j(s)x}\,, & x \le 0\,, \\ \sum_{j=1}^4 c_j(s)e^{r_j(s)x}\,, & x \ge 0\,, \end{cases}$$

where b_j and c_j are the undetermined coefficients—which, though constant in x, do depend on the Laplace variable s—and r_j are the roots of the characteristic equation

$$EIr^4 + \left[ms^2 + Cs + k\right] = 0\,,$$

ordered such that r_j is in the j quadrant of the complex plane. Imposing the condition that we have zero deformation in the limit as $|x|$ goes to infinity, that is (6d), we find that we are obliged to set some of the coefficients to zero. Since there is no overlap of subscripts, we relabel the coefficients for readability. We then have

$$\bar{y}(x,s) = \begin{cases} b_1(s)e^{r_1(s)x} + b_4(s)e^{r_4(s)x}\,, & x \le 0\,,\\ b_2(s)e^{r_2(s)x} + b_3(s)e^{r_3(s)x}\,, & x \ge 0\,. \end{cases} \tag{8}$$

We recover the value of these undetermined coefficients by imposing continuity of the function across the point force, that is equations (6e) and (6f). Additionally, integrating (7) across $x = 0$, we have

$$\begin{aligned} \int_{0^-}^{0^+} \left[EI\partial_x^4\bar{y} + \left[ms^2 + Cs + k\right]\bar{y}\right] dx + ky &= \int_{0^-}^{0^+} \tfrac{P}{s}\delta(x)\,dx\,,\\ EI\left[\partial_x^3\bar{y}\right]_{0^-}^{0^+} &= \frac{P}{s}\,, \end{aligned}$$

which is a discontinuity in the third derivative that manifests as the non-zero right hand side in (9d). Hence, we get the linear system

$$-b_1(s) + b_2(s) + b_3(s) - b_4(s) = 0\,, \tag{9a}$$

$$-r_1(s)b_1(s) + r_2(s)b_2(s) + r_3(s)b_3(s) - r_4(s)b_4(s) = 0\,, \tag{9b}$$

$$-r_1(s)^2b_1(s) + r_2(s)^2b_2(s) + r_3(s)^2b_3(s) - r_4(s)^2b_4(s) = 0\,, \tag{9c}$$

$$-r_1(s)^3b_1(s) + r_2(s)^3b_2(s) + r_3(s)^3(s)b_3(s) - r_4(s)^3b_4(s) = \frac{P}{EIs}\,, \tag{9d}$$

which is solved to give

$$b_j = \begin{cases} -\frac{P}{EIs}\left[\prod_{i=1,\, i\neq j}^{4} [r_j(s) - r_i(s)]\right]^{-1}, & j = 1, 4, \\ \frac{P}{EIs}\left[\prod_{i=1,\, i\neq j}^{4} [r_j(s) - r_i(s)]\right]^{-1}, & j = 2\,. \end{cases} \tag{10}$$

Equations (8) and (10) together give the solution for the problem (6) in the Laplace domain. To find the time-dependent solution, we are required to invert the Laplace transform. Due to the complexity of the Laplace domain function, even in this simple case, doing so is at best a difficult process. In the following section, we apply an implementation of Week's method to recover the time-dependent behaviour and compare it to the steady-state solution.

4 Results

In this section, we apply Weeks' method to the problem described in Section 3. In particular, we plot the dynamic behaviour of the beam, with a view of demonstrating the convergence of Weeks' method with increasing order N. Notably, we do not provide a verification that it converges on the correct short-term behaviour, preferring to defer this to a rigorous treatment in a future publication. We do, however, provide a verification of the validity of this solution in terms of its match to a steady-state solution as demonstrated by Sun and Luo [19]. Further, we choose to plot the solutions found with parameters matching the default parameters used for Sun and Luo's steady-state solution [19] (see Table 1). While we omit it here for brevity, the non-dimensionalisation of the governing equation (6) when considered in steady-state, can be reduced to a system with no parameters. Consequently, changes to the parameters in Table 1 effect only a scaling of the solution. While this may impact the minutia of convergence rates for Weeks' method, the broader behaviours will remain unchanged.

Table 1: Default parameters as used by Sun and Luo [19].

Parameter	Default value
P	10^4 N
EI	2.3×10^3 N/m^2
k	6.9×10^7 N/m^2
m	48 kg/m
C	10^7 Ns/m^2

Figures 1(a)–4(a) show that the solution converges and the beam settles quickly into a steady-state. Initial oscillatory behaviour observed most prominently in Figure 1 is gradually eliminated as the order of approximation N increases. This suggests that approximation with Weeks' method may not be satisfactory if sensitive calculations of the vibration of beam segments are required. However, in the more general case of needing deformation profiles, we do not expect this to be problematic.

Figures 1(b)–4(b) confirm that the steady-state deformation to which the solution by Weeks' method converges is indeed the steady-state of the system. The agreement at time $t = 5$ s appears to be very good across all orders of approximation, however, this is dependent on the particular time being compared for the lower orders (in view of the oscillations in Figures 1(a)–4(a)). To this end, while we do not provide a verification of the short-term behaviour, the solution by Weeks' method is capable at least of capturing the asymptotic behaviour of the system.

5 Conclusion

With a view to develop more complicated models, this article provided a summary of some practical considerations for the implementation of Weeks' method for numerical Laplace inversion and demonstrated its application to a beam problem. In particular, preliminary results were presented for the deformation of an Euler–Bernoulli beam on a viscoelastic foundation

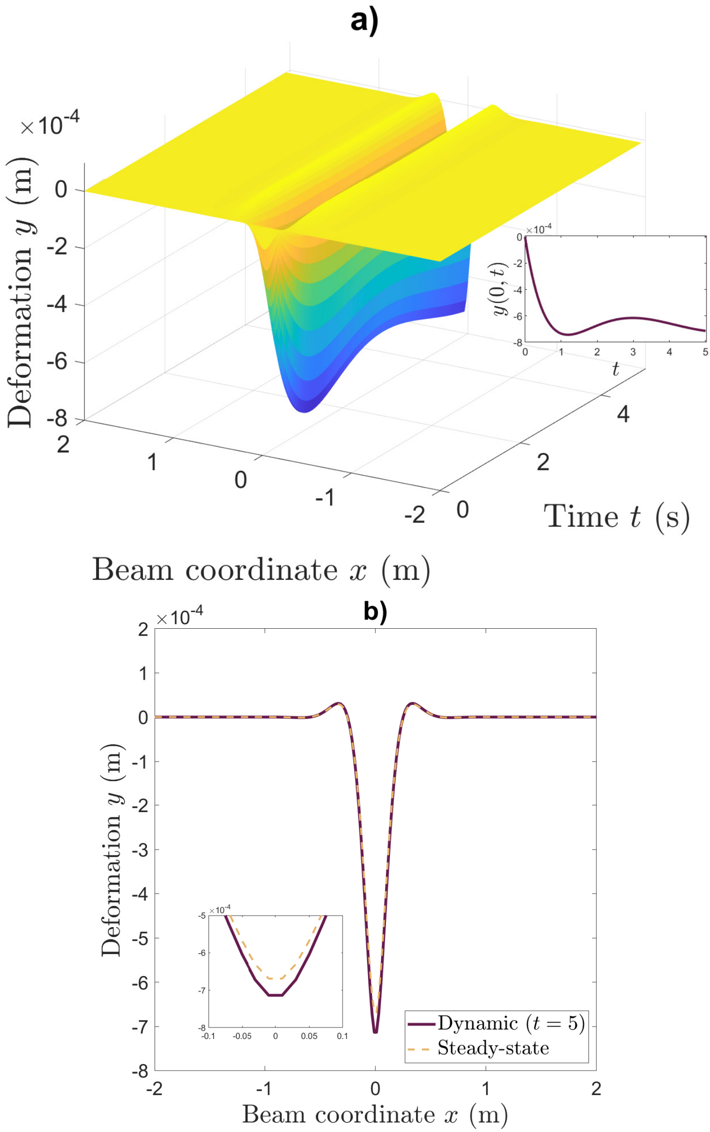

Figure 1: Solution to (6) using default values from Table 1 and Weeks' method with order $N = 16$. a) shows the dynamic solution, and b) shows a comparison between the dynamic solution at $t = 5$ and the steady-state solution [19].

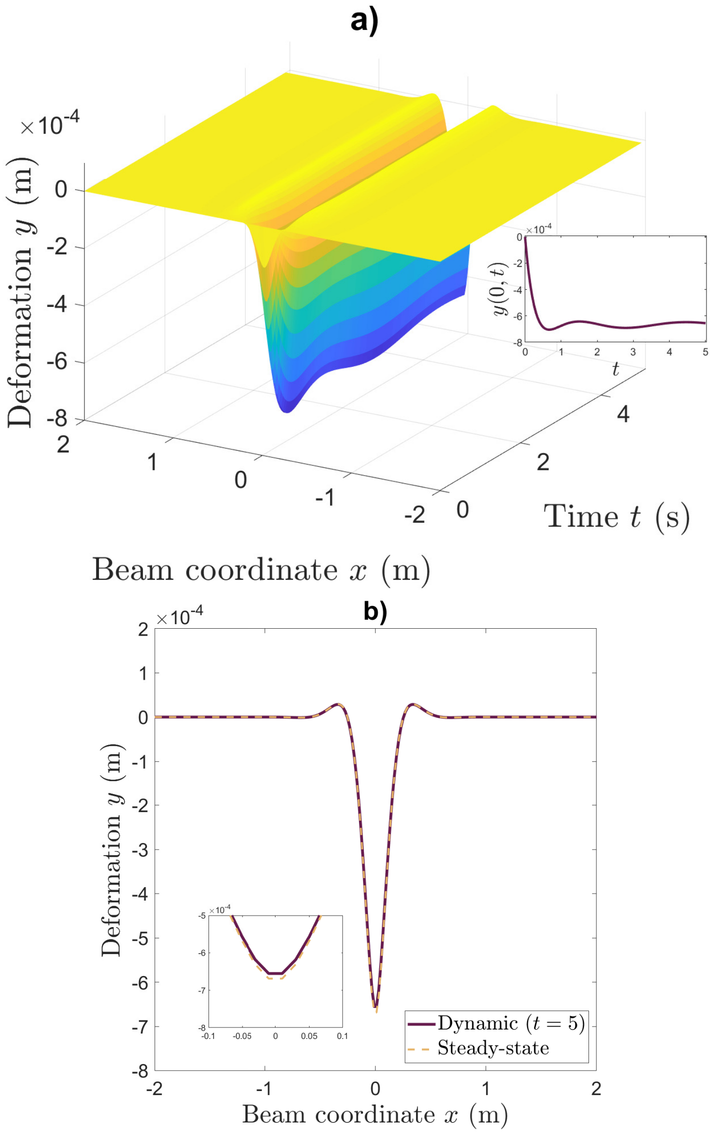

Figure 2: Solution to (6) using default values from Table 1 and Weeks' method with order $N = 32$. a) shows the dynamic solution, and b) shows a comparison between the dynamic solution at $t = 5$ and the steady-state solution [19].

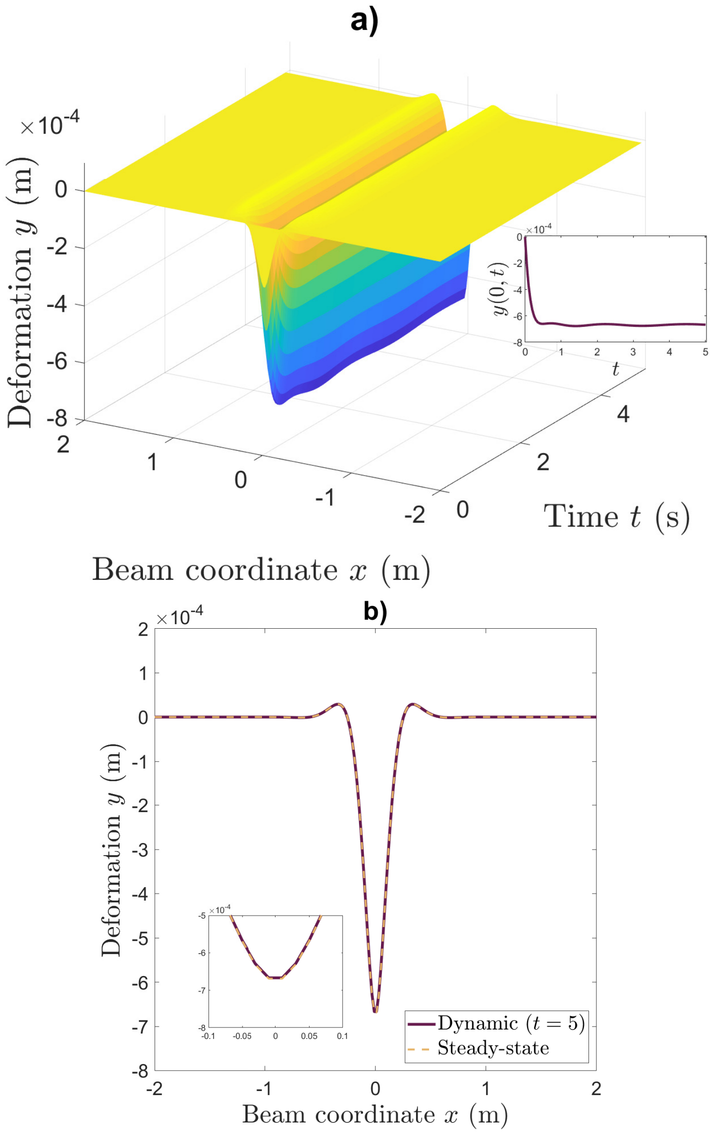

Figure 3: Solution to (6) using default values from Table 1 and Weeks' method with order $N = 64$. a) shows the dynamic solution, and b) shows a comparison between the dynamic solution at $t = 5$ and the steady-state solution [19].

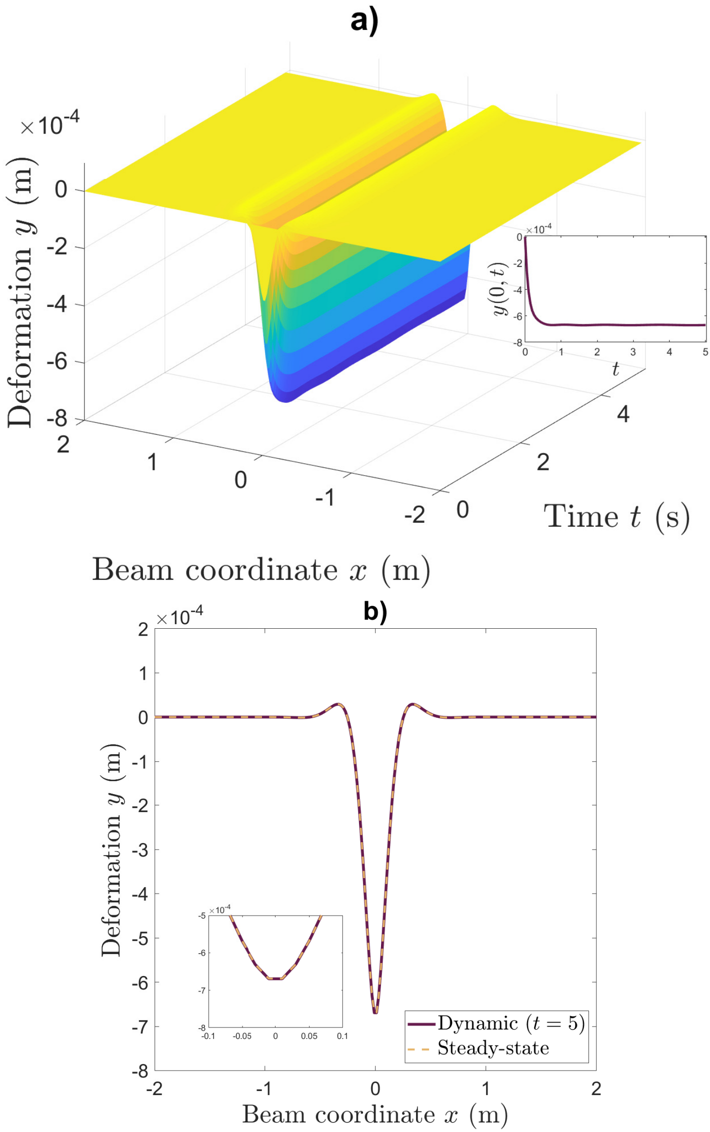

Figure 4: Solution to (6) using default values from Table 1 and Weeks' method with order $N = 128$. a) shows the dynamic solution, and b) shows a comparison between the dynamic solution at $t = 5$ and the steady-state solution [19].

subject to a stationary point force. While the short-term behaviour is left unverified (deferring this to a rigorous treatment in a future work) the long-term behaviour is verified against a published steady-state solution and found to converge as expected. The method in this article, whilst only presented for a simple model, may be extended to any model whose Laplace domain differential equation in x—the equivalent to equation (7)—may be easily solved by the method of undetermined coefficients. This is especially relevant for transition zones, which may be modelled by making the foundation coefficients, C and k, piecewise constant.

References

[1] J. Abate and W. Whitt. "A unified framework for numerically inverting Laplace transforms". In: *INFORMS J. Comput.* 18 (2006), pp. 408–421. DOI: 10.1287/ijoc.1050.0137 (cit. on p. C98).

[2] C. W. Clenshaw. "A note on the summation of Chebyshev series". In: *Math. Comput.* 9 (1955), pp. 118–120. DOI: 10.1090/S0025-5718-1955-0071856-0 (cit. on p. C102).

[3] A. M. Cohen. *Numerical methods for Laplace transform inversion.* 1st ed. Vol. 5. Boston, MA: Springer US, 2007. DOI: 10.1007/978-0-387-68855-8 (cit. on pp. C99, C102).

[4] B. Davies and B. Martin. "Numerical inversion of the Laplace transform: a survey and comparison of methods". In: *J. Comput. Phys.* 33 (1979), pp. 1–32. DOI: 10.1016/0021-9991(79)90025-1 (cit. on p. C99).

[5] D. G. Duffy. "On the numerical inversion of Laplace transforms: Comparison of three new methods on characteristic problems from applications". In: *ACM Trans. Math. Softw.* 19 (1993), pp. 333–359. DOI: 10.1145/155743.155788 (cit. on pp. C98, C99).

[6] B. S. Garbow, G. Giunta, J. N. Lyness, and A. Murli. "Software for an implementation of Weeks' method for the inverse Laplace transform". In: *ACM Transactions on Mathematical Software* 14 (1988), pp. 163–170. DOI: 10.1145/45054.45057 (cit. on pp. C99, C100).

[7] M. Guerrieri. *Fundamentals of railway design*. 1st ed. Cham: Springer Nature Switzerland, 2023, pp. 1–264. DOI: 10.1007/978-3-031-24030-0 (cit. on p. C98).

[8] B. Indraratna, P. Baral, Y. Qi, N. T. Ngo, C. Rujikiatkamjorn, and F. B. Ferreira. "Advances in ground improvement and principles of track geomechanics for future railways". In: *17th African Regional Conference on Soil Mechanics and Geotechnical Engineering*. Ed. by S. W. Jacobsz. 2020, pp. 1–16. URL: https://www.issmge.org/publications/publication/advances-in-ground-improvement-and-principles-of-track-geomechanics-for-future-railways (cit. on p. C97).

[9] B. Indraratna, Y. Qi, R. S. Malisetty, S. K. Navaratnarajah, F. Mehmood, and M. Tawk. "Recycled materials in railroad substructure: an energy perspective". In: *Railway Eng. Sci.* 30 (2022), pp. 304–322. DOI: 10.1007/s40534-021-00267-6 (cit. on p. C97).

[10] B. Indraratna, M. B. Sajjad, T. Ngo, A. Gomes Correia, and R. Kelly. "Improved performance of ballasted tracks at transition zones: A review of experimental and modelling approaches". In: *Transport. Geotech.* 21, 100260 (2019). DOI: 10.1016/j.trgeo.2019.100260 (cit. on p. C98).

[11] B. Indraratna, Q. Sun, N. T. Ngo, and C. Rujikiatkamjorn. "Current research into ballasted rail tracks: model tests and their practical implications". In: *Aust. J. Struct. Eng.* 18 (2017), pp. 204–220. DOI: 10.1080/13287982.2017.1359398 (cit. on p. C97).

[12] Bureau of Infrastructure and Transport Research Economics (BITRE). *Trainline 10*. Tech. rep. Canberra, 2023. URL: https://www.bitre.gov.au/publications/2023/trainline-10 (cit. on p. C97).

[13] K. L. Kuhlman. "Review of inverse Laplace transform algorithms for Laplace-space numerical approaches". In: *Numer. Algor.* 63 (2013), pp. 339–355. DOI: 10.1007/s11075-012-9625-3 (cit. on pp. C99, C102).

[14] M. Labrado Palomo, F. Roca Barceló, F. Ribes Llario, and J. Real Herráiz. "Effect of vehicle speed on the dynamics of track transitions". In: *J. Vib. Cont.* 24 (2017), pp. 5118–5128. DOI: 10.1177/1077546317745254 (cit. on p. C98).

[15] J. N. Lyness and G. Giunta. "A modification of the Weeks method for numerical inversion of the Laplace transform". In: *Math. Comput.* 47 (1986), pp. 313–322. DOI: 10.2307/2008097 (cit. on pp. C99, C100, C101).

[16] R. Piessens and M. A. M. Branders. "Numerical inversion of the Laplace transform using generalised Laguerre polynomials". In: *Proc. Instit. Elec. Eng.* 118 (1971), pp. 1517–1522. DOI: 10.1049/piee.1971.0278 (cit. on p. C99).

[17] M. B. Sajjad, B. Indraratna, T. Ngo, R. Kelly, and C. Rujikiatkamjorn. "A computational approach to smoothen the abrupt stiffness variation along railway transitions". In: *J. Geotech. Geoenviron. Eng.* 149, 04023063 (2023). DOI: 10.1061/JGGEFK.GTENG-11294. (Cit. on p. C98).

[18] Y. Shang, M. Nogal, R. Teixeira, and A. R. M. Wolfert. "Optimal design of rail level crossings and associated transition zones using adaptive surrogate-assisted optimization". In: *Eng. Struct.* 282, 115740 (2023). DOI: 10.1016/j.engstruct.2023.115740 (cit. on p. C98).

[19] L. Sun and F. Luo. "Steady-state dynamic response of a Bernoulli–Euler beam on a viscoelastic foundation subject to a platoon of moving dynamic loads". In: *J. Vib. Acoust.* 130, 051002 (2008). DOI: 10.1115/1.2948376 (cit. on pp. C106, C107, C108, C109, C110, C111).

[20] W. T. Weeks. "Numerical inversion of Laplace transforms using Laguerre functions". In: *J. ACM* 13 (1966), pp. 419–429. doi: 10.1145/321341.321351 (cit. on pp. C99, C100).

[21] T. Xin, Y. Ding, P. Wang, and L. Gao. "Application of rubber mats in transition zone between two different slab tracks in high-speed railway". In: *Construct. Build. Mat.* 243, 118219 (2020). doi: 10.1016/j.conbuildmat.2020.118219 (cit. on p. C98).

Author addresses

1. **J. Murray**, School of Information and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
   mailto:josiah.murray@uon.edu.au
   orcid:0000-0002-8060-911X
2. **R. Pethiyagoda**, School of Information and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
3. **M. H. Meylan**, School of Information and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
4. **N. Thamwattana**, School of Information and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia