Vector variational-like inequalities with η-generally convex mappings

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(Received 23 December 2006; revised 17 November 2007)

Abstract

Because of applications in optimization problems, mathematical programming, equilibrium problem and operations research, considerable progresses have been achieved in both theory and applications of vector variational-like inequalities. In this work, we consider vector variational-like inequalities with η-generally convex mappings and prove some existence results for our inequalities in the setting of Hausdorff topological vector space. The results presented in this article are more general and can be used to solve many known problems related to vector variational inequalities, variational-like inequalities and vector variational-like inequalities.

Contents

See http://anziamj.austms.org.au/ojs/index.php/ANZIAMJ/article/view/201 for this article, © Austral. Mathematical Soc. 2007. Published December 7, 2007. ISSN 1446-8735

1 Introduction

The vector variational inequalities in a finite-dimensional Euclidean space was first introduced by Giannessi [6], which is the vector-valued version of the variational inequality of Hartman and Stampacchia [7]. Many authors studied several kinds of vector variational inequalities in abstract spaces [2, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Ansari [1] introduced and considered vector variational-like inequalities in the setting of reflexive Banach spaces. Since then, Lee et al. [12] and Siddiqi et al. [15] have studied various kind of vector variational-like inequalities in different directions. In this work, we consider vector variational-like inequalities with η-generally convex mappings and prove some existence results for our inequalities in the setting of Hausdorff topological vector spaces.

2 Preliminaries

Definition 1 *The Hausdorff topological vector space* Y *is said to be an ordered space denoted by* (Y, P) *if ordering relations are defined in* Y *by a closed convex cone* P *of* Y *as follows:*

$$\text{for all} \quad x, y \in Y, \quad y \leqq x \Leftrightarrow x - y \in P\,,$$
$$\text{for all} \quad x, y \in Y, \quad y \leq x \Leftrightarrow x - y \in P \backslash \{0\}\,,$$

$$\text{for all} \quad x, y \in Y, \quad y \nleq x \Leftrightarrow x - y \notin P \backslash \{0\}.$$

If the interior of P, $\operatorname{int} P$, is nonempty, then the weak ordering relations in Y are also defined as follows:

$$\text{for all} \quad x, y \in Y, \quad y < x \Leftrightarrow x - y \in \operatorname{int} P,$$
$$\text{for all} \quad x, y \in Y, \quad y \not< x \Leftrightarrow x - y \notin \operatorname{int} P.$$

Let X and Y be the real Hausdorff topological vector spaces and $L(X, Y)$ be the set of all linear continuous functions from X to Y. For $l \in L(X, Y)$, the value of the linear function l at x is denoted by $\langle l, x \rangle$. Let X^* be the dual space of X and Y^* be the dual space of Y.

Throughout this article, unless otherwise specified, we assume that (Y, P) is an ordered Hausdorff topological vector space with $\operatorname{int} P \neq \emptyset$.

Definition 2 *Let E be a subset of a topological vector space X. A multivalued mapping $A : E \to 2^X$ is said to be a* KKM *mapping if, for each finite subset $\{x_1, \ldots, x_n\}$ of E, co$\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^n A(x_i)$, where co A denotes the convex hull of the set A.*

The following KKM theorem plays a crucial role in deriving existence results for our problem.

Theorem 3 (KKM–Fan Theorem) *Let E be a subset of a Hausdorff topological vector space X, let $A : E \to 2^X$ be a* KKM *map. If for each $x \in E$, $A(x)$ is closed, and if for at least one point $x \in E$, $A(x)$ is compact, then $\cap_{x \in E} A(x) \neq \emptyset$.*

Let X and Y be real Hausdorff topological vector spaces and K a subset of X. Let $T : X \to L(X, Y)$ and $\eta : K \times K \to K$, then we introduce the following vector variational-like inequality problem:

$$\begin{aligned} &\text{find } x_0 \in K \text{ such that for each } z \in K,\ \lambda \in (0, 1] \\ &\langle T(\lambda x_0 + (1 - \lambda) z), \eta(y, x_0) \rangle \notin -\operatorname{int} P, \quad \text{for all } y \in K. \end{aligned} \tag{1}$$

2.1 Special cases

1. If $\lambda = 1$, then problem (1) reduces to the vector variational-like inequality problem introduced and studied by Siddiqi et al. [13]

$$\begin{aligned} &\text{find } x_0 \in K \text{ such that} \\ &\langle Tx_0, \eta(y, x_0)\rangle \notin -\operatorname{int} P, \quad \text{for all } y \in K. \end{aligned} \tag{2}$$

2. If $\eta(y, x_0) = y - x_0$, then problem (1) is equivalent to the following vector variational inequality problem introduced and studied by Yu et al. [18]

$$\begin{aligned} &\text{find } x_0 \in K \text{ such that for each } z \in K,\ \lambda \in (0, 1] \\ &\langle T(\lambda x_0 + (1-\lambda)z),\, y - x_0\rangle \notin -\operatorname{int} P, \quad \text{for all } y \in K, \end{aligned}$$

where $T : X \to L(X, Y)$ is a mapping and $K - K \subset K$.

Definition 4 *Let* $T : X \to L(X, Y)$ *and* $\eta : K \times K \to K$. *Then* T *is said to be*

1. η-*monotone, if*

$$\langle Tx - Ty, \eta(x, y)\rangle \in P, \quad \textit{for all } x, y \in K$$

2. η-*hemicontinuous, if for all* $x, y \in K$, *the function* $t \mapsto \langle T(ty + (1 - t)x, \eta(x, y)\rangle$ *is continuous at* 0^+.

3. η-*generally convex if for any* $x, y, w, z \in K$,

$$\langle Tz, \eta(x, w)\rangle \notin -\mathit{int}\, P, \quad \textit{and} \quad \langle Tz, \eta(y, w)\rangle \notin -\mathit{int}\, P$$

implies

$$\langle Tz, \eta(\lambda x + (1-\lambda)y, w)\rangle \notin -\mathit{int}\, P, \ \textit{for all } \lambda \in [0, 1].$$

3 Existence theorems

We begin with the following lemma which is necessary for the proof of main results of this article.

Lemma 5 *Let* X *be a topological vector space,* K *a closed convex subset of* X *and* (Y, P) *be an ordered topological vector space with* $\mathrm{int}\, P \neq \emptyset$. *Let* $T : X \to L(X, Y)$ *be* η*-monotone and* η*-hemicontinuous mapping. Let* $\eta : K \times K \to K$ *be continuous and affine such that* $\eta(x, x) = 0$*, for all* $x \in K$. *Then for each* $z \in K$, $\lambda \in (0, 1]$ *the following statements are equivalent.*

Find $x_0 \in K$ *such that*

(i) $\langle T(\lambda x_0 + (1-\lambda)z), \eta(y, x_0)\rangle \notin -\mathrm{int}\, P, \quad$ *for all* $y \in K$*;*

(ii) $\langle T(\lambda y + (1-\lambda)z), \eta(y, x_0)\rangle \notin -\mathrm{int}\, P, \quad$ *for all* $y \in K$.

Proof: Let $z \in K$, $\lambda \in (0, 1]$. We denote $T_z(x) = T(\lambda x + (1-\lambda)z)$, where $x \in K$. For every $x, y \in K$, because T is η-monotone, we have

$$\frac{1}{\lambda}\langle T(\lambda x + (1-\lambda)z) - T(\lambda y + (1-\lambda)z)\,,$$
$$\eta(\lambda x + (1-\lambda)z,\ \lambda y + (1-\lambda)z) \in P\,,$$

since $\eta(\cdot, \cdot)$ is affine and $\eta(z, z) = 0$, we have

$$\langle T_z(x) - T_z(y), \eta(x, y)\rangle = \frac{1}{\lambda}\langle T(\lambda x + (1-\lambda)z) - T(\lambda y + (1-\lambda)z)\,,$$
$$\eta(\lambda x + (1-\lambda)z,\ \lambda y + (1-\lambda)z)\rangle \in P\,.$$

Hence $T_z : K \to L(X, Y)$ is also η-monotone.

Now we prove that the conclusion (i)⇒(ii). Let x_0 be a solution of (i). Since T_z is η-monotone, $\eta(\cdot,\cdot)$ is affine and $\eta(x,x)=0$,

$$\langle T_z(y) - T_z(x_0), \eta(y,x_0)\rangle \in P, \quad \text{for all } y \in K.$$

That is,

$$\langle T_z(x_0), \eta(y,x_0)\rangle \in \langle T_z(y), \eta(y,x_0)\rangle - P, \quad \text{for all } y \in K. \tag{3}$$

Suppose to the contrary that (ii) were false. Then there exists $y_0 \in K$ such that

$$\langle T(\lambda y_0 + (1-\lambda)z), \eta(y_0,x_0)\rangle \in -\text{int}\, P,$$

that is,

$$\langle T_z(y_0), \eta(y_0,x_0)\rangle \in -\text{int}\, P.$$

By (3) we obtain

$$\langle T_z(x_0), \eta(y_0,x_0)\rangle \in \langle T_z(y_0), \eta(y_0,x_0)\rangle - P.$$

Therefore

$$\langle T_z(x_0), \eta(y_0,x_0)\rangle \in -\text{int}\, P - P \subset -\text{int}\, P,$$

which contradicts (i).

Conversely, suppose that (ii) holds. Then $x_0 \in K$ satisfies

$$\langle T(\lambda y + (1-\lambda)z), \eta(y,x_0)\rangle \notin -\text{int}\, P, \quad \text{for all } y \in K.$$

That is

$$\langle T_z(y), \eta(y,x_0)\rangle \notin -\text{int}\, P, \quad \text{for all} \quad y \in K.$$

For each $y \in K$, $t \in (0,1)$, we let $y_t = ty + (1-t)x_0$. Since K is convex, thus $y_t \in K$. Then we have

$$\langle T_z(y_t), \eta(y_t,x_0)\rangle \notin -\text{int}\, P.$$

Since $\eta(\cdot,\cdot)$ is affine and $\eta(x_0, x_0) = 0$, we have

$$\langle T(\lambda(ty + (1-t)x_0) + (1-\lambda)z),\ t\eta(y, x_0)\rangle \notin -\mathrm{int}\, P\,.$$

That is

$$\langle T(\lambda(x_0 + t(y - x_0)) + (1-\lambda)z),\ \eta(y, x_0)\rangle \notin -\mathrm{int}\, P\,.$$

Considering T is η-hemicontinuous, let $t \to 0^+$, we have

$$\langle T(\lambda x_0 + (1-\lambda)z),\ \eta(y, x_0)\rangle \notin -\mathrm{int}\, P\,, \quad \text{for all } y \in K\,.$$

This completes the proof. ♠

By Lemma 5, we obtain the following theorem.

Theorem 6 *Let X be a real Hausdorff topological vector space and let K be a compact and convex subset of X, and (Y, P) be an ordered topological vector space with int $P \neq \emptyset$. Let $T : X \to L(X, Y)$ be η-monotone and η-hemicontinuous. Let $\eta : K \times K \to K$ be a continuous affine mapping such that $\eta(x, x) = 0$, for all $x \in K$. Then, problem (1) is solvable, that is, for every $z \in K$, $\lambda \in (0, 1]$, there exists $x_0 \in K$ such that*

$$\langle T(\lambda x_0 + (1-\lambda)z,\ \eta(y, x_0)\rangle \notin -\mathit{int}\, P\,, \quad \textit{for all } y \in K\,.$$

Proof: For $y \in K$, we define

$$A_1(y) = \{x \in K : \langle T(\lambda x + (1-\lambda)z),\ \eta(y, x)\rangle \notin -\mathrm{int}\, P\}$$
$$A_2(y) = \{x \in K : \langle T(\lambda y + (1-\lambda)z),\ \eta(y, x)\rangle \notin -\mathrm{int}\, P\}$$

The proof is divided into the following three steps.

1. $A_1 : K \to 2^K$ is a KKM-mapping:

   Since $y \in A_1(y)$, $A_1(y) \neq \emptyset$. Assume that there exists a finite subset $\{y_1, \ldots, y_n\} \subset K$, and $t_i \geq 0$, $i = 1, \ldots, n$ with $\sum_{i=1}^n t_i = 1$, such that

   $$x = \sum_{i=1}^n t_i y_i \notin \bigcup_{i=1}^n A_1(y_i).$$

   Clearly, $x \notin A_1(y_i)$, $i = 1, \ldots, n$. We have

   $$\langle T(\lambda x + (1-\lambda)z), \eta(y_i, x)\rangle \in -\mathrm{int}\, P\,, \quad i = 1, \ldots, n\,.$$

   Then $\langle T(\lambda x + (1-\lambda)z), \eta(y_i, \sum_{i=1}^n t_i y_i)\rangle$

   $$= \sum_{i=1}^n t_i \langle T(\lambda x + (1-\lambda)z), \eta(y_i, y_i)\rangle \in -\mathrm{int}\, P\,.$$

   Since $\eta(\cdot,\cdot)$ is affine and $\eta(y_i, y_i) = 0$, we have $0 \in -\mathrm{int}\, P$, which is a contradiction.

   Hence A_1 is a KKM mapping.

2. $\bigcap_{y\in K} A_1(y) = \bigcap_{y\in K} A_2(y)$.
   If $x \in A_1(y)$, then

   $$\langle T(\lambda x + (1-\lambda)z), \eta(y, x)\rangle \notin -\mathrm{int}\, P\,.$$

   Since T is η-monotone, $\eta(\cdot,\cdot)$ is affine and $\eta(z,z) = 0$, T_z is also η-monotone. We have

   $$\langle T(\lambda y + (1-\lambda)z) - T(\lambda x + (1-\lambda)z), \eta(y, x)\rangle \in P\,.$$

   That is,

   $$\langle T(\lambda x + (1-\lambda)z), \eta(y, x)\rangle \in \langle T(\lambda y + (1-\lambda)z), \eta(y, x)\rangle - P\,.$$

Suppose that $x \notin A_2(y)$, we have

$$\langle T(\lambda y + (1-\lambda)z), \eta(y,x)\rangle \in -\text{int}\, P\,.$$

or

$$\langle T(\lambda x + (1-\lambda)z), \eta(y,x)\rangle \in -\text{int}\, P - P \subset -\text{int}\, P\,,$$

which contradicts $x \in A_1(y)$.

Therefore, $x \in A_2(y)$, that is, $A_1(y) \subset A_2(y)$. Then

$$\bigcap_{y\in K} A_1(y) \subset \bigcap_{y\in K} A_2(y).$$

On the other hand, suppose that $x \in \bigcap_{y\in K} A_2(y)$. We have

$$\langle T(\lambda y + (1-\lambda)z), \eta(y,x) \notin -\text{int}\, P\,, \quad \text{for all } y \in K\,.$$

By Lemma 5, we obtain

$$\langle T(\lambda x + (1-\lambda)z), \eta(y,x) \notin -\text{int}\, P\,, \quad \text{for all } y \in K\,.$$

That is, $x \in \bigcap_{y\in K} A_1(y)$.

Hence $\bigcap_{y\in K} A_1(y) \supset \bigcap_{y\in K} A_2(y)$. So, $\bigcap_{y\in K} A_1(y) = \bigcap_{y\in K} A_2(y)$.

3. $\bigcap_{y\in K} A_2(y) \neq \emptyset\,.$

Since $y \in A_2(y)$, $A_2(y) \neq \emptyset\,.$ By 2, we know $A_1(y) \subset A_2(y)$. By 1, we know that A_1 is a KKM mapping. Then A_2 is also a KKM mapping.

Now, we prove that for any $y \in K$, $A_2(y)$ is closed-valued. Assume that there exists a net $\{x_n\} \subset A_2(y)$ such that $x_n \to x \in K$. Because

$$\langle T(\lambda y + (1-\lambda)z), \eta(y,x_n)\rangle \notin -\text{int}\, P, \quad \text{for all } n,$$

we have

$$\langle T(\lambda y + (1-\lambda)z), \eta(y,x)\rangle \notin -\text{int}\, P\,.$$

Hence $x \in A_2(y)$.

It follows from the compactness of K and closedness of $A_2(y) \subset K$, that $A_2(y)$ is compact. By the KKM theorem, we have $\bigcap_{y \in K} A_2(y) \neq \emptyset$, and also $\bigcap_{y \in K} A_1(y) \neq \emptyset$.

Hence there exists

$$x_0 \in \bigcap_{y \in K} A_1(y) = \bigcap_{y \in K} A_2(y);$$

that is, there exists $x_0 \in K$ such that

$$\langle T(\lambda x_0 + (1-\lambda)z), \eta(y, x_0)\rangle \notin -\mathrm{int}\, P, \quad \text{for all } y \in K,$$

that is, x_0 is the solution of the problem (1).

♠

The following Theorem 7 is proved in a different setting than Theorem 2.1 of Ansari [1]. We take $T : X \to L(X, Y)$ to be η-monotone, η-hemicontinuous and η-generally convex, although Ansari [1] considered T to be η-pseudomonotone and V-hemicontinuous.

Theorem 7 *Let X be a reflexive Banach space, (Y, P) an ordered topological vector space with $\mathrm{int}\, P \neq \emptyset$. Let K be a nonempty, bounded and convex subset of X. Let $\eta : K \times K \to K$ be a continuous and affine such that $\eta(x, x) = 0$ for all $x \in K$. Let $T : X \to L(X, Y)$ be η-monotone, η-hemicontinuous and η-generally convex on K. Then problem (1) is solvable.*

Proof:

$$A_1(y) = \{x \in K : \langle T(\lambda x + (1-\lambda)z), \eta(y, x)\rangle \notin -\mathrm{int}\, P\},$$
$$A_2(y) = \{x \in K : \langle T(\lambda y + (1-\lambda)z), \eta(y, x)\rangle \notin -\mathrm{int}\, P\},$$

where $y, z \in K$, $\lambda \in (0, 1]$.

By using the proof of Theorem 6, we know that A_2 is a KKM mapping and $A_2(y)$ is closedly valued. We also know that

$$\bigcap_{y \in K} A_1(y) = \bigcap_{y \in K} A_2(y).$$

Because K is bounded, closed, convex and X is a reflexive Banach space, therefore K is weakly compact.

Now, we prove that $A_2(y)$ is convex. Suppose that $y_1, y_2 \in A_2(y)$ and $t_1, t_2 \geq 0$ with $t_1 + t_2 = 1$. Then

$$\langle T(\lambda y + (1-\lambda)z, \eta(y, y_i)\rangle \notin -\text{int}\, P\,, \quad i = 1, 2\,.$$

Since T is η-generally convex,

$$\langle T(\lambda y + (1-\lambda)z, \eta(y, t_1 y_1 + t_2 y_2)\rangle \notin -\text{int}\, P\,,$$

that is, $t_1 y_1 + t_2 y_2 \in A_2(y)$. Hence $A_2(y)$ is convex. Since $A_2(y)$ is closed and convex, $A_2(y)$ is weakly closed.

Considering that $A_2(y)$ is a KKM mapping and $A_2(y)$ is a weakly closed subset of K, $A_2(y)$ is weakly compact. By using the KKM theorem, there exists $x_0 \in K$ such that $x_0 \in \bigcap_{y \in K} A_1(y) = \bigcap_{y \in K} A_2(y) \neq \emptyset$. That is, there exists $x_0 \in K$ such that

$$\langle T(\lambda x_0 + (1-\lambda)z, \eta(y, x_0)\rangle \notin -\text{int}\, P\,, \quad \text{for all} \quad y \in K\,.$$

Hence problem (1) is solvable. ♠

References

[1] Q. H. Ansari, On generalized vector variational-like inequalities, *Ann. Sci. Math. Quebec* **19**(1995), 131–137. E34, E42

[2] G. Y. Chen, Existence of solution for a vector variational inequality; An extension of the Hartman-Stampacchia theorem, *J. Optim. Theory Appl.* **74**(1992), 445–456. E34

[3] G. Y. Chen and G.H. Cheng, *Vector Variational Inequality and Vector Optimization Problem*, In *Lecture Notes in Economics and Mathematical Systems*, volume 258, Springer–Verlag, (1987). E34

[4] G. Y. Chen and B.D. Craven, A vector variational inequality and optimization over an efficient set, *Zeitscrift fur Operations Research* **3**(1990), 1–12. E34

[5] G. Y. Chen and X.Q. Yang, Vector complementarity problems and its equivalences with weak minimal element in ordered spaces, *J. Math. Anal. Appl.* **153**(1990), 136–158. E34

[6] F. Giannessi, *Theorems of Alternative, Quadratic Programs and Complementarity Problems*, Edited by R. W. Cottle, F. Giannessi and J. L. Lions, John Wiley and Sons, Chichester, 151–186, 1980. E34

[7] P. Hartman and G. Stampacchia, On some nonlinear elliptic differential functional equations, *Acta Math.* **115**(1966), 271–310. E34

[8] I. V. Konnov and J. C. Yao, On the generalized vector variational inequality problem, *J. Math. Anal. Appl.* **206**(1997), 42–58. E34

[9] B. S. Lee and G. M. Lee, A vector version of Minty's lemma and application, *Appl. Math. Lett.* **12**(5)(1999), 43–50. E34

[10] G. M. Lee, D. S. Kim, B. S. Lee and N. D. Yen, Vector variational inequality as a tool for studying vector optimization problems, *Nonlinear Analysis* **34**(1998), 745–765. E34

[11] G. M. Lee, B. S. Lee and S. S. Chang, On vector quasi-variational inequalities, *J. Math. Anal. Appl.* **203**(1996), 626–638. E34

[12] B. S. Lee, G. M. Lee and D. S. Kim, Generalized vector variational-like inequalities on locally convex Hausdorff topological vector spaces, *Indian J. Pure Appl. Math.* **28**(1997), 33–41. E34

[13] S. Park, B. S. Lee and G. M. Lee, A general vector-valued variational inequality and its fuzzy extension, Internat. J. Math. & Math. Sci. **21**(4)(1998), 637–642. E34, E36

[14] S. K. Mishra and M. A. Noor, On vector variational-like inequality problems, *J. Math. Anal. Applics.* **311**(1)(2005), 69–75. E34

[15] A. H. Siddiqi, Q. H. Ansari and R. Ahmad, On vector variational-like inequalities, *Indian J. Pure Appl. Math.* **28**(8)(1997), 1009–1016. E34

[16] A. H. Siddiqi, Q. H. Ansari and A. Khaliq, On vector variational inequalities, *J. Optim. Theory Appl.* **84**(1)(1995), 171–180. E34

[17] X. Q. Yang, Vector variational inequality and its duality, *Nonlinear Analysis, T. M. A.* **21**(1993), 869–877. E34

[18] S. J. Yu and J. C. Yao, On vector variational inequalities, *J. Optim. Theory Appl.* **89**(1996), 749–769. E34, E36

[19] X. M. Yang and X. Q. Yang, Vector variational-like inequality with pseudoinvexity, *Optimization* **55**(1-2)(2006), 157–170. E34

[20] Yali Zhao and Zunquan Xia, On the existence of solutions to generalized vector variational-like inequalities, *Nonlinear Analysis* **64**(9)(2006), 2075–2083. E34

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