Periodic wave solution of a second order nonlinear ordinary differential equation by Homotopy analysis method

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Abstract

The periodic wave solution of a second order nonlinear ordinary differential equation is obtained by the homotopy analysis method, an analytical, totally explicit mathematical technique. By choosing a proper auxiliary parameter, the new series solution converges very fast. The method provides us with a simple way to adjust the convergence region. Furthermore, a significant improvement of the convergence rate and region is achieved by applying Homotopy-Padé approximants. Three examples demonstrate the excellent computation accuracy and efficiency of the present HAM approach. The present method could be extended for more complicated wave equations.

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1 Introduction

Nonlinear problems are of interest to many scientists and engineers because most physical systems in the real world are inherently nonlinear in nature. There are many analytical techniques for decomposing a nonlinear equation into a set of linear equations, such as perturbation method, Lyapunov's artificial small parameter method, the δ-expansion method and Adomian's decomposition method. However, most of these analytical techniques cannot provide us with a convenient way to control and adjust the convergence region and rate of solution series. For some strong nonlinear problems, these analytical approaches may break down. Recently, an analytical technique, namely homotopy analysis method (HAM) has seen rapid development. It has many successful applications in nonlinear problems and logically contains Lyapunov's small parameter method, the δ-expansion method, and Adomian's decomposition method [3]. The HAM does not depend on a small parameter such as in a perturbation approach, and has great flexibility in the selection of a proper set of base functions for the solution. The method also provides a simple way to control the convergence rate and region. The HAM was first applied to the fluid mechanics problem by Liao [2] and has been systematically described by Liao [3]. Liao and Cheung [4] successfully applied HAM to nonlinear waves propagating in deep water and the HAM solution in finite water depth was later obtained by Tao et al. [5]. Abbasbandy [1] and Wang et al. [6], also obtained periodic wave solutions by HAM technique for third-order shallow water wave equations.

The HAM is applied to solve a second order nonlinear differential equation with periodic boundary conditions. The equation has some physical prototypes, such as Drinfeld–Sokolov–Wilson wave equation and modified Zakharov–Kuznetsov equation. Multiple periodic wave solutions are obtained by different sets of base functions. Explicit solutions are presented and conditions for the existence of the periodic wave solutions are given. Very accurate solutions are obtained with only the first few terms of the series, demonstrating the high efficiency of HAM.

2 Theoretical consideration

Consider a second order nonlinear ordinary differential equation with periodic boundary conditions in the following form

$$w''(x) + \alpha w(x) + \beta w^3(x) = 0\,, \quad w(0) = w(L) = 0\,, \tag{1}$$

where x is a spatial variable, $w(x)$ is a real function of x, and the prime denotes the differentiation. For Drinfeld–Sokolov–Wilson wave equation, $\alpha > 0$ and $\beta < 0$, whilst for modified Zakharov–Kuznetsov equation, $\alpha < 0$ and $\beta > 0$.

Under the transformation

$$x = (L/\pi)\tau\,, \quad \epsilon = (L/\pi)^2, \quad v(\tau) = w(x)\,, \tag{2}$$

equation (1) becomes

$$v''(\tau) + \epsilon\alpha v(\tau) + \epsilon\beta v^3(\tau) = 0\,, \quad v(0) = v(\pi) = 0\,. \tag{3}$$

Supposing $A = v(\pi/2\kappa)$ is the amplitude of the wave and κ is a positive integer, we define a new function

$$u(\tau) = v(\tau)/A\,, \quad u(\pi/2\kappa) = 1\,. \tag{4}$$

Equation (1) then finally becomes

$$u''(\tau) + \epsilon \left[\alpha u(\tau) + A^2\beta u^3(\tau)\right] = 0, \quad u(0) = u(\pi) = 0\,. \tag{5}$$

Equation (5) has multiple solutions. It is natural to express periodic wave solution of $u(\tau)$ by the set of base functions

$$\{\sin[(2m+1)\kappa\tau] \mid m, \kappa = 1, 2, 3, \ldots\}\,, \tag{6}$$

in the form

$$u(\tau) = \sum_{m=1}^{+\infty} b_m \sin[(2m+1)\kappa\tau]\,, \tag{7}$$

where b_m are coefficients to be determined.

HAM is based on a continuous variation from an initial trial to the exact solution. By constructing the homotopic mapping $u(\tau) \longrightarrow U(\tau; q)$, we have the following homotopy

$$(1-q)\mathcal{L}[U(\tau;q) - u_0(\tau;q)] = q\hbar\mathcal{N}\left[U(\tau;q), A(q)\right]\,, \tag{8}$$

subject to the boundary conditions

$$U(0;q) = U(\pi;q) = U(\pi/2\kappa;q) = 0\,. \tag{9}$$

where $U(\tau;q)$ is differentiable with respect to the embedding parameter q, $A(q)$ is the mapping function of A, $u_0(\tau)$ is an initial estimate of $u(\tau)$, $\hbar$ is a nonzero auxiliary parameter, $\mathcal{N}$ is a nonlinear operator, and $\mathcal{L}$ is a linear auxiliary operator with the property $\mathcal{L}[0] = 0$.

When $q = 0$ and $q = 1$, then

$$U(\tau;0) = u_0(\tau)\,, \tag{10}$$

$$\mathcal{N}[U(\tau;1)] = 0\,, \tag{11}$$

respectively. Therefore, as the embedding parameter q varies from 0 to 1, $U(\tau;q)$ maps continuously from the initial estimate $u_0(\tau)$ to the exact solution $u(\tau)$.

Equation (8) is named the *zeroth order deformation equation* in HAM. In this article, the nonlinear operator $\mathcal{N}$ is chosen as

$$\mathcal{N}\left[U(\tau;q),A(q)\right] = \frac{\partial^2 U(\tau;q)}{\partial\tau^2} + \kappa\left[\alpha U(\tau;q) + \beta A^2(q)U^3(\tau;q)\right] . \qquad (12)$$

The linear operator $\mathcal{L}$ is a linear auxiliary operator with the property $\mathcal{L}[0] = 0$, which is chosen as

$$\mathcal{L}\left[U(\tau;q)\right] = \left(\frac{\partial^2}{\partial\tau^2} + \kappa^2\right) U(\tau;q)\,, \qquad (13)$$

with the property

$$\mathcal{L}\left[C_1\sin(\kappa\tau) + C_2\cos(\kappa\tau)\right] = 0\,, \qquad (14)$$

where C_1 and C_2 are coefficients.

According to the boundary condition (5) and the *rule of solution expression* (7), the initial guess is chosen as

$$u_0(\tau) = \sin(\kappa\tau)\,. \qquad (15)$$

Expand $U(\tau;q)$ and $A(q)$ in Taylor series with respect to q, we have

$$U(\tau;q) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)q^m\,, \qquad (16)$$

$$A(q) = a_0 + \sum_{m=1}^{+\infty} a_m q^m\,, \qquad (17)$$

where

$$u_m(\tau) = \frac{1}{m!}\left.\frac{\partial^m U(\tau;q)}{\partial q^m}\right|_{q=0}\,, \qquad (18)$$

$$a_m = \frac{1}{m!}\left.\frac{\partial^m A(q)}{\partial q^m}\right|_{q=0}\,. \qquad (19)$$

If the series (16) converges at $q = 1$, we have the solution

$$U(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)\,. \tag{20}$$

For brevity, we define

$$\vec{u}_m = \{u_0, u_1, u_2, \ldots, u_m\}, \tag{21}$$
$$\vec{a}_m = \{a_0, a_1, a_2, \ldots, a_m\}\,. \tag{22}$$

Differentiating equations (8) and (9) m times with respect to q at $q = 0$, and then dividing them by $m!$, the mth order deformation equation is

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar R_m(\vec{u}_{m-1}, \vec{a}_{m-1})\,, \tag{23}$$

subject to the boundary conditions

$$u_m(0) = u_m(\pi) = u_m(\pi/2\kappa) = 0\,, \tag{24}$$

where

$$\begin{aligned} R_m(\vec{u}_{m-1}, \vec{a}_{m-1}) &= \frac{1}{(m-1)!} \left.\frac{\partial^{m-1}\mathcal{N}[U(\tau;q)]}{\partial q^{m-1}}\right|_{q=0} \\ &= u''_{m-1}(\tau) + \epsilon\alpha u_{m-1}(\tau) \\ &\quad + \epsilon\beta \sum_{n=0}^{m-1}\sum_{i=0}^{n} a_i a_{n-i}\left[\sum_{j=0}^{m-1-n} u_j(\tau) \sum_{r=0}^{m-1-n-j} u_r(\tau) u_{m-1-n-j-r}(\tau)\right]. \end{aligned} \tag{25}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1\,, \\ 1, & m > 1\,, \end{cases} \tag{26}$$

The right-hand side of equation (23) is expressed as

$$\hbar R_m(\vec{\phi}_{m-1}, \vec{a}_{m-1}) = \sum_{n=0}^{\mu_m} b_{m,n}(\vec{a}_{m-1}) \sin[(2n+1)\kappa\tau]\,, \tag{27}$$

where $b_{m,n}(\vec{a}_{m-1})$ is a coefficient depending on terms up to $(m-1)$th order and μ_m is a positive integer.

The term $b_{m,0}(\vec{a}_{m-1})$ must be zero, otherwise the solution of equation (23) contains the term $\tau \sin(\kappa\tau)$, which disobeys the *rule of solution expression* (7). This provide us an equation to determine a_{m-1}.

The solution of equation (23) is

$$\begin{aligned} u_m(\tau) &= \chi_m u_{m-1}(\tau) \sum_{n=1}^{\mu_m} \frac{b_{m,n}}{1-(2n+1)^2\kappa^2} \sin[(2n+1)\kappa\tau] \\ &\quad + C_1 \sin(\kappa\tau) + C_2 \cos(\kappa\tau) \,, \end{aligned} \tag{28}$$

According to the boundary condition (24) and *rule of solution expression* (7), we have

$$C_2 = 0 \,, \tag{29}$$

and C_1 is determined by the equation $u_m(\pi/2\kappa) = 0$.

3 Result and discussion

From the equation $b_{1,0}(\vec{a}_0) = 0$, we have the solution

$$a_0 = 2\sqrt{\frac{\kappa^2 - \alpha\epsilon}{3\beta\epsilon}} \,, \tag{30}$$

So for different values of α and β the periodic wave solution exists only at

$$\begin{cases} \kappa \geqslant \sqrt{\alpha\epsilon}\,, & \text{if } \alpha > 0 \text{ and } \beta > 0\,, \\ \text{any } \kappa, & \text{if } \alpha < 0 \text{ and } \beta > 0\,, \\ \kappa \leqslant \sqrt{\alpha\epsilon}\,, & \text{if } \alpha > 0 \text{ and } \beta < 0\,, \\ \text{no } \kappa, & \text{if } \alpha < 0 \text{ and } \beta < 0\,. \end{cases} \tag{31}$$

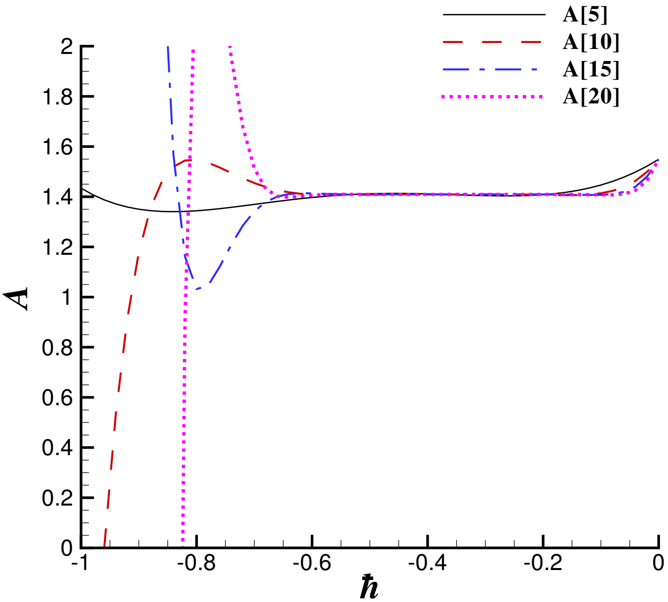

Figure 1: The 5th, 10th, 15th and 20th order approximations of A versus $\hbar$.

The convergence region and rate are controlled by the auxiliary parameter $\hbar$ in HAM. For different values of $\hbar$, A converges to the same value—the approximation of the exact solution. As shown in Figure 1, the nearly horizontal line segments of A-$\hbar$ curves correspond to the convergence regions of the $\hbar$ values. Figure 1 clearly shows that the HAM approximations of the amplitude A converge in a region around $\hbar \in [-1/2, -1/5]$. Therefore, the auxiliary parameter is chosen as $\hbar = -1/2$ for all the HAM solutions presented in this section.

The zeroth, first and second order approximations of $u(\tau)$ and A are

$$U_0(\tau) = \sin(\kappa\tau)\,, \tag{32}$$

$$u_1(\tau) = \left[1 + \frac{1}{24}\hbar - \frac{\hbar\alpha\epsilon}{24\kappa^2}\right]\sin(\kappa\tau) + \left[\frac{1}{24}\hbar - \frac{\hbar\alpha\epsilon}{24\kappa^2}\right]\sin(3\kappa\tau)\,, \tag{33}$$

$$\begin{aligned} u_2(\tau) = &\left[1 + \frac{1}{12}\hbar + \frac{11}{288}\hbar^2 - \frac{\hbar^2\alpha^2\epsilon^2}{288\kappa^4} - \frac{\hbar\alpha\epsilon}{12\kappa^2} - \frac{5\hbar^2\alpha\epsilon}{144\kappa^2}\right]\sin(\kappa\tau) \\ &+ \left[\frac{1}{12}\hbar + \frac{23}{576}\hbar^2 - \frac{\hbar^2\alpha^2\epsilon^2}{576\kappa^4} - \frac{\hbar\alpha\epsilon}{12\kappa^2} - \frac{11\hbar^2\alpha\epsilon}{288\kappa^2}\right]\sin(3\kappa\tau) \\ &+ \left[\frac{1}{576}\hbar^2 + \frac{\hbar^2\alpha^2\epsilon^2}{576\kappa^4} - \frac{\hbar^2\alpha\epsilon}{288\kappa^2}\right]\sin(5\kappa\tau)\,, \end{aligned} \tag{34}$$

$$A_0 = 2\sqrt{\frac{\kappa^2 - \alpha\epsilon}{3\beta\epsilon}}\,, \tag{35}$$

$$A_1 = \left[\frac{2\hbar\alpha\epsilon\kappa^2 - \hbar\alpha^2\epsilon^2 - \hbar\kappa^4}{24\left(\kappa^2 - \alpha\epsilon\right)} + 2\right]\sqrt{\frac{\kappa^2 - \alpha\epsilon}{3\beta\epsilon}}\,, \tag{36}$$

$$\begin{aligned} A_2 = &\left[\frac{384\hbar\alpha\epsilon\kappa^4 + 171\hbar^2\alpha\epsilon\kappa^4 - 7\hbar^2\alpha^3\epsilon^3 - 192\hbar\alpha^2\epsilon^2\kappa^2 - 75\hbar^2\alpha^2\epsilon^2\kappa^2}{2304\kappa^4\left(\kappa^2 - \alpha\epsilon\right)}\right. \\ &\left. - \frac{192\hbar\kappa^6 + 89\hbar^2\kappa^6}{2304\kappa^4\left(\kappa^2 - \alpha\epsilon\right)} + 2\right]\sqrt{\frac{\kappa^2 - \alpha\epsilon}{3\beta\epsilon}}\,. \end{aligned} \tag{37}$$

Choosing the first three terms of series (16) and (17) respectively and using the built-in function *PadeApproximant* in *Mathematica 6*, the $[1,1]$ homotopy-Padé (HP) approximation of $u(\tau)$ is

$$\begin{aligned} u_{[1,1]}(\tau) = &-\frac{\sqrt{\kappa^2 - \alpha\epsilon}}{288\kappa^4\sqrt{3\beta\epsilon}}\sin(\kappa\tau)\left[\hbar\alpha\epsilon - (\hbar - 48)\kappa^2\right] \\ &\times\left[\hbar\alpha\epsilon - (12 + \hbar)\kappa^2 + \hbar\left(\alpha\epsilon - \kappa^2\right)\cos(2\kappa\tau)\right]\,. \end{aligned} \tag{38}$$

The first $2m$th order solutions and $[m,m]$ homotopy-Padé approximations of A are shown in Table 1. The series converges very quickly, especially

Table 1: The $2m$th order solutions and $[m, m]$ homotopy-Padé approximations of A for $\alpha = 2$, $\beta = -1$, $\epsilon = 5$ and $\kappa = 2$.

Order	A	$[m, m]$	A
2	1.23635	[1, 1]	1.22928
4	1.23095	[2, 2]	1.22993
6	1.23005	[3, 3]	1.22992
8	1.22993	[4, 4]	1.22992
10	1.22992	[5, 5]	1.22992

for the homotopy-Padé approximation. The $[2, 2]$ HP approximation gives the same result as the eighth order solution, a clear demonstration of the excellent convergence rate in the present homotopy-Padé technique.

Figure 2, 3 and 4 are three examples of $v(\tau)$ $(= Au(\tau))$ for different types of combinations of the three parameters:

Figure 2 $\alpha = 2$, $\beta = -1$ and $\epsilon = 5$;

Figure 3 $\alpha = -1$, $\beta = 2$ and $\epsilon = 2$;

Figure 4 $\alpha = 2$, $\beta = 2$ and $\epsilon = 4$.

For the first example, there are only three periodic wave solutions, that is, $\kappa = 1$, $\kappa = 2$ and $\kappa = 3$. As the parameter κ increases, the amplitude of the waves decrease and the crest of the wave becomes sharper. For the second example, the number of the periodic wave solutions is infinite and the parameter κ can be any positive integer. In contrast to the first example, the amplitude of the periodic waves increase as the parameter κ rises. For the third example, the number of the periodic wave solutions is infinite as well. However, parameter κ must be greater than two. The amplitude of the periodic waves is also a monotonically increasing function of the parameter κ as in the second example.

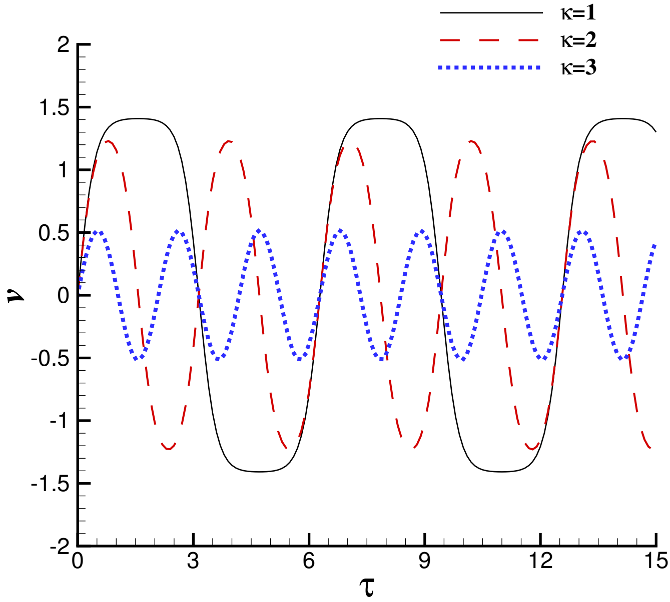

Figure 2: The tenth order solution of $\nu(\tau)$ for $\alpha = 2$, $\beta = -1$ and $\epsilon = 5$.

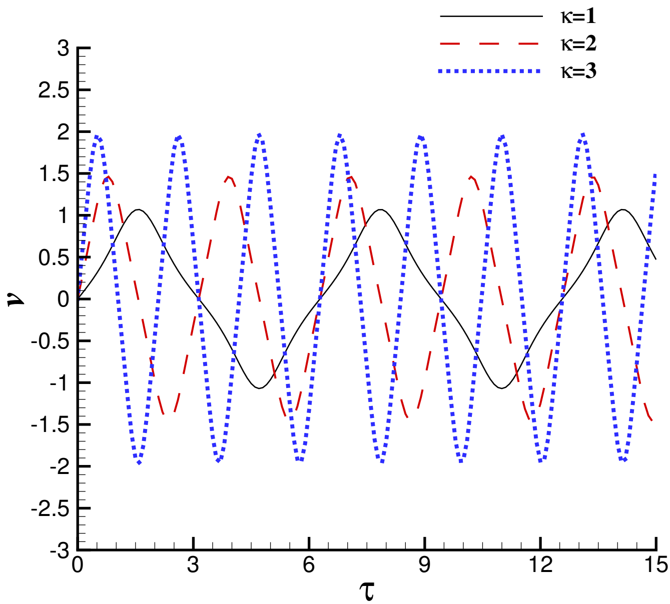

Figure 3: The tenth order solution of $\nu(\tau)$ for $\alpha = -1$, $\beta = 2$ and $\epsilon = 2$.

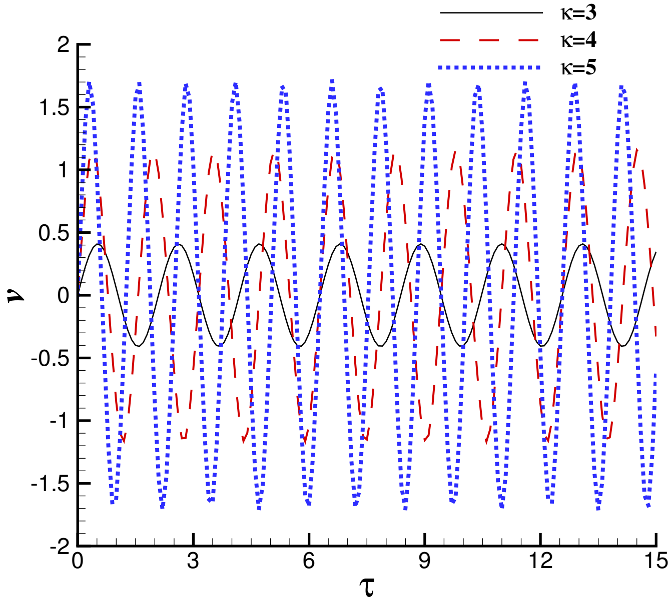

Figure 4: The tenth order solution of $\nu(\tau)$ for $\alpha = 2$, $\beta = 2$ and $\epsilon = 4$.

4 Conclusion

Explicit periodic wave solutions of a second order nonlinear ordinary differential equation with arbitrary parameters were solved by the homotopy analysis method. Multiple solutions were given by different sets of base functions and the bifurcation points were found. The convergence region is controlled by the non-zero parameter $\hbar$, providing us with a simple way to adjust convergence. Furthermore, a significant improvement of the convergence rate and region is achieved by applying homotopy-Padé techniques. The present method could be extended to provide periodic wave solutions for more complicated wave equations.

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