A simple battle model with explanatory power

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Abstract

Attrition equations have military use and are also used in biological and economic modelling. We model the aggregation of attrition in a battle to explain the strong support in historical data for the log law, which conventionally is thought to apply mainly to losses through accident or illness. Support for the log law has been found in many studies of battle data and this has yet to be explained. Several historical studies found support for a mixture of attrition laws, suggesting that different laws could apply to different parts of the battle. We hypothesise that the log law could be supported through aggregation effects when other laws apply on a micro scale. We assume that all laws work at skirmish level and show that aggregation effects will only support the log law if the individual skirmishes being aggregated are themselves modelled by the log law. We argue that the extreme support for the log law in the Kursk dataset is due to an overwhelming support for that law at the level of individual skirmishes, and that the conventional use of square and linear law for skirmishes is incorrect. These results suggest that theoretical changes to attrition equations should be based on studies of small unit attrition as aggregation effects do not cause cross over from square or linear laws to log law.

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1 Introduction

There are many models of military conflict, with individual models varying in the level of detail required and in the generality of their outcomes. At the most abstract and generalised level, Lanchester–Ozipov models [7] have been strongly used for more than ninety years: for example, a review [13] to 1993 listed over 700 articles, most of which were theoretical developments. Yet these models have never been empirically validated for battles which are aggregations of shorter engagements.

Lanchester–Ozipov models are coupled first order differential equations for force attrition, and are instantiations of

$$\frac{dx}{dt} = -f(x, y, v), \qquad (1)$$

where x and y are the numbers in the X and Y forces at time t and v is a vector of other variables, such as contextual and tactical variables, and where the right-hand side f incorporates interactions between variables. The instantiations are the square, linear and log laws, where the square law has

$$f(x, y, \nu) = k_1(\nu) y \,, \tag{2}$$

with the attrition rate a multiple of the number of enemy soldiers. The linear law has

$$f(x, y, \nu) = k_2(\nu) x y \,, \tag{3}$$

where there is difficulty in acquiring targets or the firing is unaimed and attrition depends equally on the number of shooters and the number of targets. The log law is based on empirical data [8] and has

$$f(x, y, \nu) = k_3(\nu) x \,, \tag{4}$$

where the attrition rate is completely independent of the number in the enemy force. The coefficient $k_i(\nu)$ is constant for the event being modelled.

There are two major problems for Lanchester equations which arise from historical battle data [2, 5, 6]. First, the three Lanchester–Ozipov laws do not explain much of the variation in daily attrition. Second, the log law has an unexpected and unexplained prominence. In a companion article [9] we searched the Kursk dataset [4] for information to explain the prominence of the log law. Speight [12] suggested that the art of war, in the form of force balancing by commanders, artificially caused the log law to be prominent. We found significant collinearity between the daily numbers in the two forces, which supports the existence of force balancing. However, collinearity interferes with statistical discrimination and, to minimise the interference, we analysed the data using force ratios and loss ratios. The results unequivocally supported the log law and rejected the square and linear laws; a remarkable outcome which requires explanation.

We show that the Kursk dataset supports a pure log law. We also develop an aggregation model to determine if square law and linear law skirmishes aggregate to give an apparent support for the log law and use it to show that the aggregate attrition will support a pure log law only if the great majority of the skirmishes also follow the log law.

We follow the historical approach in using a continuous model for a discrete variable (attrition). The model is essentially statistical, with expected values being calculated.

2 Aggregation effects

Land warfare is complex. Helmbold [3] (1964) stated that (historical data) "suggests that victory in battle is primarily determined by factors other than numerical superiority". The complexity is due to many factors, including contextual variables such as terrain, weather, engineering works, the mix and numbers of different types of entities on each side, the actions or engagements being undertaken by entities, and that a battle is the aggregation of all of the engagements [2, 6, 12]. Context can change within an individual engagement [12], so we define skirmishes which may be a fraction of an engagement, in time and space, for which the context is constant. We first show that the daily attrition for the Kursk dataset [4] displays an apparent linear dependence on the force ratio, then we remove this dependence, developing a test for any residual relationship between daily attrition and force ratio. We show that there is no residual connection for the Kursk dataset so that the aggregate trend at Kursk is a pure log law, not merely in the log-linear spectrum. Then we build the aggregation model to investigate ways that a pure log law can be formed for the aggregate attrition, showing that it can be formed only if the skirmishes are also pure log laws.

2.1 Log law trend in data and a test for underlying dependence

Under a pure square law dependence, the loss ratio $\Delta y/\Delta x$ would be proportional to the inverse force ratio x/y. For pure linear law dependence, the loss ratio would be constant. For pure log law the loss ratio would be proportional to the force ratio y/x. Some analyses, both empirical and via simulation, have reported losses which have a dependence in between the square law and the log law; that is, with the power of x somewhere between 1 and -1 and the power of y between -1 and 1. We show that, for the Kursk dataset, the dependence of loss ratio on force ratio is pure log law. In Figure 1 the daily loss ratio is plotted against the daily force ratio for the fourteen days. The graph shows a distinct proportionality between the two variables, with the trend appearing to cut the axes very close to the origin.

If the trend is purely log law, it must follow a relationship

$$\frac{\Delta y}{\Delta x} = F\frac{y}{x}\,, \tag{5}$$

where F is independent of the force ratio and nominally constant although it varies from day to day due to other factors. Using the fractional losses for the two forces, we construct the fractional loss ratio

$$\frac{\Delta y/y}{\Delta x/x} = F\,. \tag{6}$$

If the relationship between loss ratio and force ratio is not purely a log law, then the fractional loss ratio will show a residual dependence on the force ratio.

There is no such residual relationship, as seen from the scatter diagram in Figure 2. The correlation between the two variables is almost zero (0.036) and the probability that it is zero is $p = 0.90$.

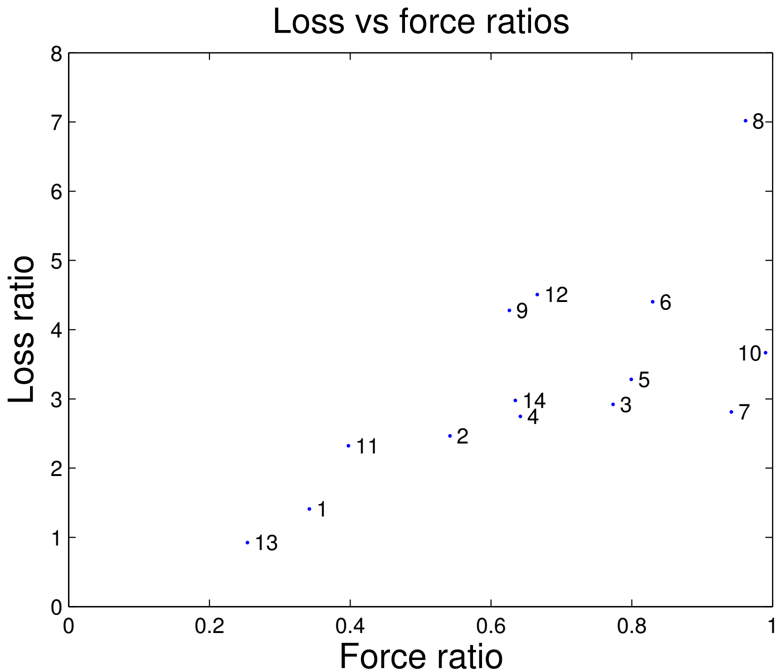

FIGURE 1: Loss ratio as a function of force ratio using daily Kursk data. Points annotated by day number.

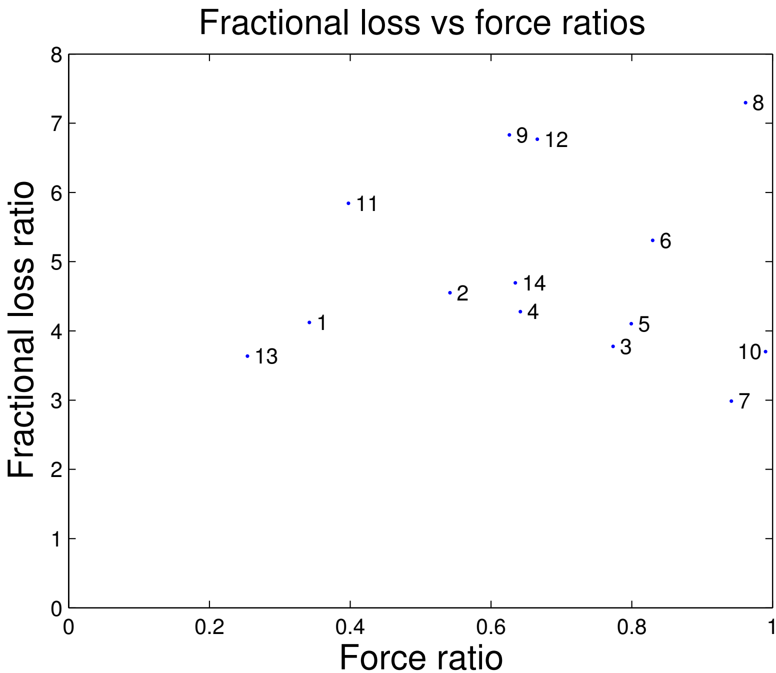

Figure 2: Fractional loss ratio as a function of force ratio for daily Kursk data. Points are annotated by day number. Force ratio calculated from the number in each force who were active on that day

2.2 The basic aggregation model

We represent each force as a set of small groups, where the groups each act as a unit for the entire day. Force X has N_x groups where the numbers of people are represented by $\{a_{ix}, i = 1, \ldots, N_x\}$. Similarly, for force Y, the groups are $\{a_{jy}, j = 1, \ldots, N_y\}$. The only restriction on group size is that $1 \leqslant a_{ix} \leqslant x_0$ and $1 \leqslant a_{jy} \leqslant y_0$ where x_0 is the total number in the active X force at the beginning of the day and y_0 is the equivalent for force Y. We next represent attrition for the day as the sum of attritions suffered via skirmishes between small groups. For either side, attrition is the sum of elements in the matrix $\{A_{zij},\ i = 1, \ldots, N_x,\ j = 1, \ldots, N_y\}$ where $z = x$ or $z = y$. We then model the attrition due to each skirmish, integrate over the day and aggregate over all skirmishes. Initially we do this for the case where all skirmishes are modelled by a single law.

2.3 Square law skirmishes

Under the square law the attrition for force Y through skirmishes between a_{ix} and a_{jy} during a day is

$$A_{yij} = \int C_{ix}(t) a_{ix} \, dt \,. \tag{7}$$

If a_{ix} and a_{jy} are functions of time, their values may change due to the actions of other groups during the day and the analysis will be complicated. We take a different but equivalent approach. Each group has an effectiveness at causing attrition which is some function of the number in the group. If a group suffers casualties, the group effectiveness, and C_{ix}, will be reduced. If $G_{ix}(t)$ is the group effectiveness, then $C_{ix}(t) = G_{ix}(t)/a_{ix}$ where a_{ix} is the number of members in the group at the beginning of the day. If an observer measures the value of $C_{ix}(t)$ during the day, we approximate it by a piecewise continuous function so that $C_{ix}(t)$ has the values $\{C_{1ix}, C_{2ix}, \ldots, C_{mix}\}$ for skirmishes involving group ix during the day, where each C_{kix} is true for a particular period T_{kix}. Then

$$A_{yij} = \sum_{k=1}^{m} C_{kix} T_{kix} a_{ix} = D_{ix} a_{ix} \,, \tag{8}$$

where D_{ix} is defined implicitly by this equation, and the total Y force attrition for the day is

$$A_y = \sum_{i=1}^{N_x} D_{ix} a_{ix} \,. \tag{9}$$

The weighted sum in Equation(9) also represents the dot product of the two vectors $\mathbf{D_x} = (D_{1x}, D_{2x}, \ldots)$ and $\mathbf{a_x} = (a_{1x}, a_{2x}, \ldots)$:

$$A_y = \mathbf{D_x} \bullet \mathbf{a_x} = |\mathbf{D_x}||\mathbf{a_x}| \cos\theta \,. \tag{10}$$

The expression $|\mathbf{a_x}|$ is the size of the vector $\mathbf{a_x}$ and we show the relationship between A_y and x by using an elementary relationship for variance $\sum X^2/n = \overline{X}^2 + \sigma^2$. We write

$$A_y = N_x \sqrt{(\overline{D}_x^2 + \sigma_{Dx}^2)(\overline{a}_x^2 + \sigma_{ax}^2)} \cos\theta \,, \tag{11}$$

where $\overline{D}_x$ and $\overline{a}_x$ are the means of the elements of the vectors $\mathbf{D_x}, \mathbf{a_x}$. When $\sigma_{ax} = 0$ then $N_x \sqrt{\overline{a}_x^2 + \sigma_{ax}^2} = N_x \overline{a}_x = x$ and we see that variance in group size is the only factor which can prevent the aggregation of square laws from being a square law. The elements of the vector $\mathbf{D_x}$ could vary enormously from day to day and this variation could, to some degree, mask the square law relationship.

2.4 Linear law and log law skirmishes

When skirmishes follow the linear law,

$$A_{yij} = \int C_{ij}(t) a_{ix} a_{jy} \, dt \,, \tag{12}$$

and aggregation is over the elements of a matrix. However, we apply the approach used for the square law to each row of the matrix and finally sum the sums of rows to get an equation for aggregate attrition proportional to $N_x N_y \sqrt{(\overline{a}_x^2 + \sigma_{ax}^2)(\overline{a}_y^2 + \sigma_{ay}^2)}$. Similarly, log law skirmishes aggregate to be proportional to $N_y \sqrt{\overline{a}_y^2 + \sigma_{ay}^2}$ for Y force attrition. Thus, for a force to be represented in the same way in both aggregate attrition and skirmish equations, the quadratic mean of its group sizes needs to be approximately equal to the square of the linear mean. The variation in the standard deviation for group sizes from day to day will also need to be less than any changes in aggregate force size.

2.5 Outcomes of aggregation

If skirmishes follow a single law, only the linear law might appear to support the log law: we show that even this possibility vanishes under the ratio test. The aggregate attrition A_y for the linear law skirmishes might show the proportionality $A_y \propto y$ if the variance of group sizes in the X force is such that the quadratic mean of its group sizes is not almost equal to the square of the linear mean. Our test for support for the log law uses the ratio of losses for each side. Group sizes will be identical for each of the two attrition equations, so in this case we will have $A_x \propto y$ which appears to support the square law. When the loss ratio is formed we see that it is expected to remain constant, supporting the linear law. For a single law aggregate to support the log law it is necessary that the skirmishes also support the log law.

When skirmishes follow different laws, the loss ratio takes the form

$$\frac{\Delta y}{\Delta x} = \frac{\alpha_1 x + \alpha_2 xy + \alpha_3 y}{\beta_1 y + \beta_2 xy + \beta_3 x} . \tag{13}$$

In the case of complete symmetry, the aggregate result will appear to support the linear law, otherwise the outcome will be unpredictable except when there are strong asymmetries. When one side is mainly square law and the other mainly log law, the result will again appear to support the linear law. When one side is mainly linear law and the other is mainly square or log law, the loss ratio will apparently depend on the numbers in one of the forces. In the Kursk dataset, the loss ratio is correlated with the Soviet force size ($r = 0.60$), raising the possibility that Soviet losses are the sum of linear and log laws while German losses follow the log law. However, Soviet force size correlates strongly with the force ratio ($r = 0.90$) raising distinct collinearity issues. This is resolved by correlating Soviet force numbers with the fractional loss ratio. The resulting coefficient ($r = 0.09$) shows that the Kursk data is explained by a pure log law, while the aggregation analysis shows that an aggregate log law can only be explained by log law skirmishes.

2.6 Explanations

Since the effect of artillery fire supports the log law, one explanation is that almost all of the casualties were caused by artillery. While this would be theoretically correct and is almost supported by published estimates (for example, that artillery was responsible for 60% of casualties in World War II [1]), it is hard to believe that a single technology could be responsible for, say, 90% of casualties in a battle.

The alternative is that the other causes of attrition, such as infantry and tank fire, are not correctly represented by the Lanchester–Osipov models at the skirmish level. This possibility needs to be investigated.

3 Summary

Some previous studies of historical battle data detected limited support for a mixture of three Lanchester–Osipov laws (square, linear and log), with different laws apparently describing different skirmishes and with the aggregate being somewhere in between. Support for the log law has previously been seen as an anomaly and it was proposed that collinearity between the numbers of soldiers active each day for the two sides could cause an actual support for the square and linear laws to apparently support the log law. In a companion article, we minimised the effects of collinearity for the Kursk dataset by analysing ratios of parameters for the two sides and found that the ratio data strongly supported the log law and rejected the square and linear laws. It had also been proposed that the aggregation of attrition from skirmishes could cause an apparent support for the log law.

We looked for evidence of other laws underlying the support for the log law in the Kursk dataset. The apparent proportionality between loss ratio and force ratio was removed by the calculation of the fractional loss ratio and that ratio was shown to be completely independent of the force ratio, showing that the Kursk data represented a pure log law, where the loss ratio is exactly proportional to the force ratio. The constant of proportionality showed random variation from day to day due to other factors. We also built a model of aggregate attrition, based on the assumption that Lanchester–Osipov laws work at the skirmish level. Complexities in the battlefield could interfere with the link between the skirmish model and the aggregate model. We isolated the single factor that determines if a model at skirmish level is also expressed at aggregate level and we showed that neither square nor linear law at skirmish level can cause apparent support for the log law at aggregate level. Support for the log law at aggregate level rests on support for the log law at skirmish level, and pure log law support at aggregate level implies that all skirmishes are also log law. On the basis of this evidence, the assumption of the validity of square and linear laws at skirmish level is unjustified and all mechanisms in all skirmishes need to be able to support log law attrition.

We showed that the log law is the subject of serious differences between battle models and historical battle data. Now we raise some other important modelling issues. Within the context of battle modelling, the validity of continuous approximations to a discrete variable is an open question which needs to be investigated. We also need to address possible sources of a dependence between the attrition for a force and the actions it has taken, and to investigate the influence of the stochastic nature of conflict. In earlier modelling, using Markov chains [10, 11], we showed that some stochastic variation can be accounted for by an envelope approach and that a side can limit its own attrition rate by controlling the time its forces spend in danger as well as by limiting the lethality of the enemy.

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