Euler's disk: examples used in engineering and applied mathematics teaching

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Abstract

Euler's disk is a toy described at http://www.eulersdisk.com. Aspects of its motion are modelled as an ideal disk rolling on a horizontal plane. In the final stages of Euler disk motions, the disk is nearly flat to the plane. Asymptotic approximations to the frequency of finite amplitude oscillations on steady (non-dissipative) rolling motions of the Euler disk are described. There are two different approximations which are appropriate in different limits. When the parameters are such that both apply, the formulae for the frequency agree: this appears to be new and simple. The material has been used in teaching; the teaching, and related, materials are available via the web [Keady, Math2200 Lecture Handouts, UWA].

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1 Introduction

The phenomena illustrated by Euler's disk—an abrupt end to the motion, 'stopping in finite time'—is also illustrated with a coin spinning on a horizontal surface. Although dissipative effects are crucial for the above phenomenon, there are other quantities of interest; for example, some sound frequencies associated with oscillatory non-dissipative motions that may be possible to measure experimentally. This article treats non-dissipative motions only, which are described by equations (1) below.

Equations (1) have often been used over the past century as an example in applied maths teaching [13, 10]. The Euler disk toy has been sold for just a decade. Most of the studies in recent research articles on the Euler disk have, correctly, focussed on the central unsolved problems of the dissipative mechanisms. The results in this article were largely discovered when teaching applied and engineering maths involving topics like the numerical solution of ODEs, and stability of equilibria. The Euler disk motivated our studies, studies which appear to be largely new but decidedly elementary, concerning the oscillations about the motions of the disk moving at a small angle to the 'table' (horizontal plane). In particular, the asymptotics given in equation (6) of §1.2 are new. The ODE (16) for certain motions nearly flat to the horizontal plane was new to the problem at the time this article was written. Since then Srinivasan and Ruina [8] noted a similar use for it. The analysis of the ODE (16), given in §4, is, in the context of this problem, also new. However, the ODE has other applications, and some results we found are re-discoveries of those given by Whittaker [13]. The methods we use are elementary. The material in this article, together with the supporting materials given by Keady [4], is intended to complement the existing treatments of the rolling disk problem in applied mathematics textbooks [10, e.g.].

There is room for future work in several directions. None of the asymptotics in this article have been rigorously proven. More importantly, it is not clear from the published experimental data that the nonlinear oscillations solving the ODEs are observed in the late stages of the Euler disk motions treated in this article. For references to the experimental articles, see §6.

Our definitions follow those of Synge and Griffith [10]. The radius of the disk is denoted by a and the acceleration due to gravity by g. The inclination of the plane of the disk to the vertical is θ. Figure 1 shows the rectangular coordinate system moves with the disk and with its origin C at the centre of the disk. The first coordinate axis, and associated direction $\mathbf{i}$, is in the direction of the line joining the centre of the disk to the point of contact P. The vector $\mathbf{k}$ is normal to the plane of the disk. The vector $\mathbf{j}$, so that $\mathbf{ijk}$ is an orthogonal triad, is therefore in the plane of the disk and horizontal. The angular velocity of the disk is $\boldsymbol{\omega} = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$. One finds that the four unknowns θ, ω_1, ω_2 and ω_3 satisfy a set of four, first order, ODEs. Our equations (1a) to (1d) are readily derived from these, where γ parametrises

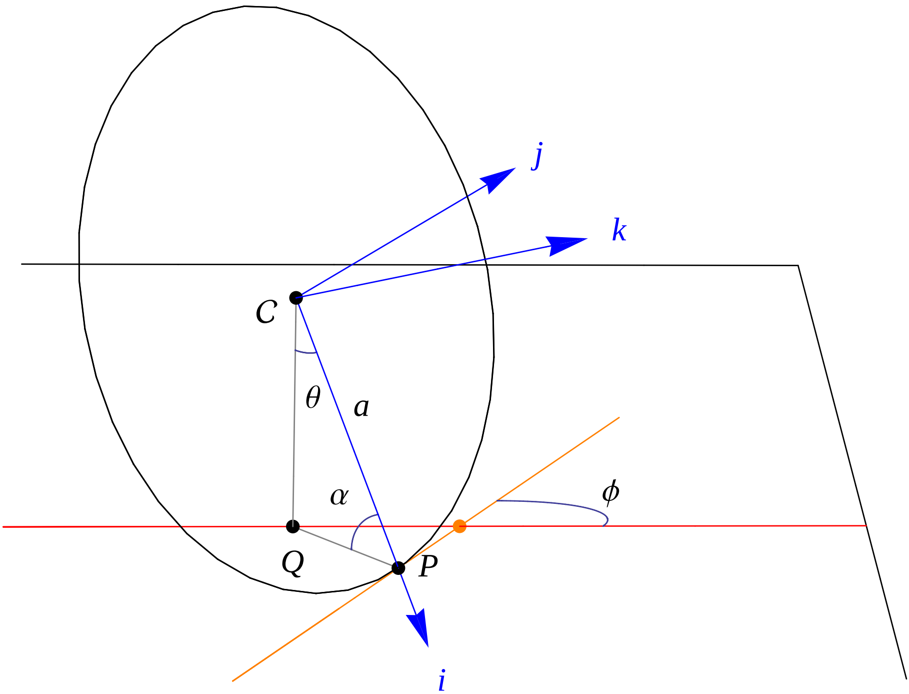

Figure 1: Disk rolling on a plane.

the mass distribution of the rotationally symmetric disk: $\gamma = 2/3$ for a uniform disk, and $1/2$ for a hoop.

Much of the interest in the Euler disk motions concerns its behaviour when the disk is nearly flat to the table; that is, θ is just a little less than $\pi/2$. Thus we define the complementary angle $\alpha = \pi/2 - \theta$, which we normally take to be small and positive. We record, from §4, at the right below, approximating equations for small α [4, Part II]:

$$\begin{aligned}
\dot\theta &= -\omega_2\,, && (1a)\\
\dot\omega_1 &= -\omega_2\left(\tan(\theta)\omega_1 + 2\omega_3\right), && (1b)\\
\dot\omega_2 &= \frac{(1-\gamma)\tan(\theta)\omega_1^2 + 2\omega_1\omega_3}{(1+\gamma)} \\
&\quad - \frac{2\gamma\sin(\theta)}{(1+\gamma)}\frac{g}{a}\,, && (1c)\\
\dot\omega_3 &= -\gamma\omega_2\omega_1\,. && (1d)
\end{aligned}$$

$$\begin{aligned}
\dot\alpha &= \omega_2\,, && (2a)\\
\dot\omega_1 &= -\omega_2\,\omega_1/\alpha\,, && (2b)\\
\dot\omega_2 &= \frac{(1-\gamma)}{(1+\gamma)}\frac{\omega_1^2}{\alpha} \\
&\quad - \frac{2\gamma}{(1+\gamma)}\frac{g}{a}\,, && (2c)\\
\dot\omega_3 &= -\gamma\omega_2\omega_1\,. && (2d)
\end{aligned}$$

Provided $\theta(0) \neq (n+1/2)\pi$, n integer, solutions to initial value problems for (1) exist, either for all time, or for an interval $[0, t_*)$ where t_* is such that $\cos(\theta(t)) \to 0$ as $t \to t_*$. This latter possibility is that of the disk falling into the 'table' (the horizontal plane).

We have $dE/dt = 0$ where the energy E is

$$4\gamma E = (1-\gamma)\omega_1^2 + (1+\gamma)\omega_2^2 + 2\omega_3^2 + 4\gamma\frac{g}{a}\cos(\theta)\,.$$

The angle ϕ, shown in Figure 1, defined by Synge and Griffith [10], satisfies $\omega_1 = -\cos(\theta)\dot\phi$.

1.1 Stability of equilibria and linear oscillations

The systems (2) and (1) are of the form $\dot{y} = f(y)$. *Equilibrium points* are vectors y_e where $f(y_e) = 0$. We assess stability by linearizing, setting up a problem for the remainder ρ where $y = y_e + \rho$, which is $\dot\rho = J\rho$, where $J = Df(y_e)$ is the Jacobian of f evaluated at y_e.

The signs of the real parts of the eigenvalues of J classify the stability of an equilibrium: if all the real parts are negative, the equilibrium is stable; if all the real parts are nonpositive, with at least one of the real parts zero, we will describe the equilibrium as neutrally stable; if any real part is positive, the equilibrium is unstable.

For any of the rolling disk equilibria, J is rank two and $J^3 = \lambda^2 J$ with λ^2 a simple expression in terms of the components of J [4, Part I], . When λ is pure imaginary, the frequency of the oscillations is then $\nu_0 = |\lambda|$.

We treat equilibria where $0 < |\theta_{eq}| < \pi/2$ with the disk rolling in a circle and ignore the degenerate case of $\theta = 0$ where the disk is either rolling in a straight line or spinning on its axis. For these equilibria, when θ_{eq} is sufficiently close to $\pi/2$, all the equilibria are (neutrally) stable. This is relevant to the smooth behaviour of the Euler disk after its short period of initial unsteady behaviour.

1.2 Nonlinear oscillations

The systems (2) and (1) are integrable. Dividing equations (1b) and (1d) by equation (1a), or dividing equations (2b) and (2d) by equation (2a), yields linear ODEs that have closed form solutions [7]. The solutions for $\omega_1(\theta)$ and $\omega_3(\theta)$ depend on the initial conditions. On substituting the expressions for $\omega_1(\theta)$ and $\omega_3(\theta)$ into equation (1c) or (2c) one finds that the variables θ and α each satisfy a ODE of the form

$$\ddot{u} + f(u) = 0\,. \tag{3}$$

A first integral of this is $\mathcal{E}$ is constant where

$$\mathcal{E}(u, \dot{u}) = \frac{1}{2}(\dot{u})^2 + F(u) \quad \text{with } F(u) = \int f(u)\, du\,. \tag{4}$$

Infinitesimally small oscillations are treated by the methods of the preceding subsection, by finding the eigenvalues of J. Applying this to the equilibria of

§2, yields the frequency

$$\nu_0^2 = \frac{g}{a}\frac{6\gamma}{1+\gamma}\frac{1}{\cos(\theta_{eq})} \sim \frac{g}{a}\frac{6\gamma}{1+\gamma}\frac{1}{\alpha_{eq}}\,, \quad \text{where } \alpha_{eq} = \pi/2 - \theta_{eq}\,. \tag{5}$$

Consider periodic solutions arising as perturbations of the equilibria of §2. We find that the frequency ν of these oscillations satisfies

$$\nu \sim \nu_0 \left[1 - \frac{5}{12}\left(\frac{\alpha_{eq} - \alpha_{min}}{\alpha_{eq}}\right)^2 + \cdots\right] \quad \text{under appropriate limits.} \tag{6}$$

In §3 the limit, denoted (L_3), is

$$\text{first } \left(\frac{\alpha_{eq} - \alpha_{min}}{\alpha_{eq}}\right) \to 0, \quad \text{then } \alpha_{eq} \to 0\,. \tag{L_3}$$

In §4 the limit, denoted (L_4), is

$$\text{first } \alpha_{eq} \to 0, \quad \text{then } \left(\frac{\alpha_{eq} - \alpha_{min}}{\alpha_{eq}}\right) \to 0\,. \tag{L_4}$$

We find that the asymptotics for the frequency in equation (6) also hold under limits (L_4).

There are many open questions, both mathematical [4, Part II] and physical (§6).

2 Moffatt's equilibria

In the simplest model of the Euler disk motion, the point of rolling contact describes a circle with a constant angular velocity, $\dot\phi$, the centroid is fixed, and the motion persists forever. These are equilibria of (1) with $\omega_2 = 0 = \omega_3$ and $\omega_1 = -\cos(\theta)\dot\phi$. The equation that says $\dot\omega_2 = 0$ (when $\sin(\theta) \neq 0$) gives

$$\omega_1^2 = \frac{g}{a}\frac{2\gamma}{1-\gamma}\cos(\theta) \quad \text{or} \quad \dot\phi^2 = \frac{g}{a}\frac{2\gamma}{(1-\gamma)\cos(\theta)}\,. \tag{7}$$

When α is small, as in the final stages of an Euler disk motion, and we deal with a disk ($\gamma = 2/3$), we have $\dot{\phi}^2 \sim 4g/(a\alpha)$ [5, equation (1)].

Approximate models, giving a first impression of what might be happening, have the centroid slowly falling vertically and energy slowly being dissipated. The expression for the energy, in the $\omega_2 = 0 = \omega_3$ equilibrium case, for a disk, is

$$E = \frac{3}{2} mga\cos(\theta) \sim \frac{3}{2} mga\alpha\,. \tag{8}$$

Moffatt's analysis [5] supposed that energy slowly dissipates, the motion being like a slow evolution along a branch of equilibria: it leads to a 'finite stopping time'. (The precise mechanisms involved in the dissipation, and whether slipping is important, remain unresolved.)

For the rest of this article there is no dissipation. As noted above, it is easy to calculate the natural frequency $\nu_0 = |\lambda|$ for the very small perturbations on the steady rolling above. For these equilibria, $\omega_3 = 0$ and α is small, and ν_0 is given by equation (5) [4]. Both ν_0 and $\dot{\phi}$ have the same asymptotic dependence on α_{eq}; that is, both are proportional to $\sqrt{g/(a\alpha_{\text{eq}})}$. For a disk, $\nu_0/\dot{\phi} = \sqrt{15}\,$.

Without dissipation, periodic perturbations of the steady rolling also persist. One can imagine such perturbations as 'rocking' about a diameter. Larger amplitude rocking perturbations are the subject of this article. With $\theta = \pi/2 - \alpha$ we are concerned with initial value problems with

$$\alpha(0) = \alpha_0\,, \quad (\text{we usually take } \alpha_0 = \alpha_{\min}\,,\ 0 < \alpha_{\min} < \alpha_{\text{eq}}) \tag{9a}$$

$$\omega_1(0) = \frac{\alpha_{\text{eq}}}{\alpha_0}\sqrt{\frac{g}{a}\,\frac{2\gamma}{(1-\gamma)}\cos(\theta_{\text{eq}})} \sim \sqrt{\frac{g}{a}\,\frac{2\gamma}{(1-\gamma)}\,\frac{\alpha_{\text{eq}}^3}{\alpha_0^2}}\,, \tag{9b}$$

$$\omega_2(0) = 0\,, \tag{9c}$$

$$\omega_3(0) = 0\,. \tag{9d}$$

3 Small nonlinear perturbations

3.1 Poincaré–Lindstedt for the ODE (3)

Let u_{eq} be a (neutrally) stable equilibrium for the ODE (3). Approximate,

$$f(u) = \nu_0^2(u - u_{eq}) + \frac{1}{2}f''(u_{eq})(u - u_{eq})^2 + \frac{1}{6}f'''(u_{eq})(u - u_{eq})^3 + \cdots,$$

where $\nu_0^2 = f'(u_{eq})$. Then, as found by a Poincaré–Lindstedt asymptotic approximation [6, §2.3.2], small oscillations about equilibrium solution satisfy

$$u \sim u_{eq} + \epsilon\cos(\nu t) + \frac{\epsilon^2 f''(u_{eq})}{4\nu_0^2}\left(\frac{1}{3}\cos(2\nu t) - 1\right) + \cdots, \tag{10}$$

$$\nu \sim \nu_0\left[1 + \frac{9(f'''(u_{eq})/6)\nu_0^2 - 10(f''(u_{eq})/2)^2}{24\nu_0^4}\epsilon^2\right] + \cdots, \tag{11}$$

as $\epsilon \to 0$.

3.2 The rolling disk in general

As described in the beginning of §1.2, we derive the following linear ODEs:

$$\frac{d\omega_1}{d\theta} = \tan(\theta)\omega_1 + 2\omega_3, \quad \frac{d\omega_3}{d\theta} = \gamma\omega_1. \tag{12}$$

These are solved [7, equation (17)] using the initial values for θ, ω_1 and ω_3 to find, on using (1c), that $\theta(t)$ satisfies a ODE of the form $\ddot\theta + f(\theta) = 0$. While the explicit formula for $f(\theta)$ is elaborate [7, equation (17)], the right-hand side of (1c) is relatively simple. Using the ODEs (12) as substitution rules, derivatives with respect to θ, that is $f^{(n)}(\theta)$, can be calculated to be of the form

$$f^{(n)}(\theta) = f_{1,1,n}\omega_1^2 + f_{1,3,n}\omega_1\omega_3 + f_{3,3,n}\omega_3^2 + f_{0,n},$$

where the $f_{*,n}$ on the right, when multiplied by $\cos^{n+1}\theta$, are polynomial, total degree at most $(n+1)$ in $\sin\theta$ and $\cos\theta$. To use this in the asymptotics (10,11) we need to make use of the initial conditions and to evaluate the derivatives at θ_{eq}.

3.3 The $\omega_3 = 0$, $\theta \neq 0$ equilibria

We treat perturbations about the equilibria described in §2 using the ODEs (1) with the initial conditions (9). To approximate the ϵ^2 term of equation (11) we need only have the leading term in the approximations:

$$\omega_3(\theta_{eq}) \sim 0\,, \quad \omega_1(\theta_{eq})^2 \sim \frac{g}{a}\frac{2\gamma}{1-\gamma}\cos(\theta_{eq})\,.$$

The zeroth order term ν_0 follows from the eigenvalues of the Jacobian. The expression for ν_2 in general is lengthy. When we take a series expansion for α_{eq} near zero, we get the asymptotics (6) under the limit (L_3).

4 Approximations with $\theta \approx \pi/2$, $\omega_3 \approx 0$

We now write $\omega_1(\alpha)$ rather than $\omega_1(\theta) = \omega_1(\pi/2-\alpha)$.

Thomson [11, p.154] notes that one can do much more than a mere linear stability analysis and that when α is small, there are significant simplifications to the ODEs describing the nonlinear motions. We approximate $\cos\theta \sim \alpha$ and $\sin\theta \sim 1$. We also follow Thomson [11] in

$$\text{assuming } \omega_3 \text{ is much smaller than } \omega_1/\alpha\,. \tag{13}$$

(Batista [1] gives a more general study, without requiring ω_3 to be small.) Using this assumption in equation (1b) for $\dot\omega_1$ we obtain (2b), which can be integrated with the initial conditions (9a–9b) to get

$$\omega_1(\alpha) \sim \frac{\omega_1(\alpha_0)\alpha_0}{\alpha}\,.$$

Using this in the $\dot\omega_3$ equation (1d), we find

$$\omega_3(\alpha) \sim \omega_3(\alpha_0) - \gamma\omega_1(\alpha_0)\alpha_0 \log\left(\frac{\alpha}{\alpha_0}\right). \qquad (14)$$

We can now check some internal consistency of the approximation. If we start with $\alpha\omega_3/\omega_1$ small, as in (9d), then

$$\frac{\alpha\omega_3}{\omega_1} \approx -\gamma\alpha^2 \log\left(\frac{\alpha}{\alpha_0}\right), \qquad (15)$$

and thus it remains small provided α_{max} is small.

On using assumption (13) in equation (1c) we derive (2c) and then find

$$\ddot\alpha = \dot\omega_2 = \frac{(1-\gamma)}{(1+\gamma)}\frac{(\omega_1(\alpha_0)\alpha_0)^2}{\alpha^3} - \frac{2\gamma}{(1+\gamma)}\frac{g}{a}\,. \qquad (16)$$

At this stage it is appropriate to state that we pose the problem with first $\alpha_{\text{eq}} > 0$ given and small, which then specifies the initial ω_1 and this, in turn, enters into the coefficients of the second order ODE. So we rewrite (16) as

$$\ddot\alpha + f(\alpha) = 0\,, \quad \text{where} \quad f(\alpha) = -\frac{\nu_0^2\alpha_{\text{eq}}}{3}\left(\frac{\alpha_{\text{eq}}^3}{\alpha^3} - 1\right). \qquad (17)$$

There is precisely one equilibrium for this ODE and it is (neutrally) stable. The rather elaborate choice in (9b) is so that the equilibrium point for our simple ODE will be the same as the α_{eq} associated with our solutions of problem (1). When (9c) is satisfied, a first integral of (17) is

$$\mathcal{E}(\alpha,\omega_2) = \frac{1}{2}\omega_2^2 + F(\alpha) = F(\alpha_0) \quad \text{where } F(\alpha) = \frac{1}{2}\frac{\nu_0^2\alpha_{\text{eq}}}{3}\left(\frac{\alpha_{\text{eq}}^3}{\alpha^2} + 2\alpha\right), \qquad (18)$$

and $\mathcal{E}$ is the energy in this approximation up to a factor of $a^2 m(1+\gamma)/2\gamma$. Define the involution

$$\mathrm{invol}_\alpha(\alpha) = \frac{\alpha_{\mathrm{eq}}^3 + \alpha_{\mathrm{eq}}^{3/2}\sqrt{8\alpha^3 + \alpha_{\mathrm{eq}}^3}}{4\alpha^2}.$$

Then

$$\alpha_{\max} = \mathrm{invol}_\alpha(\alpha_{\min}), \quad \alpha_{\min} = \mathrm{invol}_\alpha(\alpha_{\max}).$$

A 'difficulty' with Problem (1) is that there are solutions [3] which 'fall through the table'. In this regard, our second order ODE (17) is a lot simpler. Note: (a) $\mathcal{E}(\alpha, \omega_2) \to \infty$ as α tends down to 0 from above; and (b) $\mathcal{E}(\alpha, \omega_2)$ is a convex function in the half space $\{(\alpha, \omega_2) \mid \alpha > 0\}$.

Theorem 1

1. *Any solution of the ODE (17) which starts in the positive half-space $\{(\alpha, \omega_2) \mid \alpha > 0\}$ of the phase space remains in that half-space.*
2. *The trajectory, in the (α, ω_2) phase space, of any solution of the differential equation (17) which has $\alpha(0) > 0$ is a closed convex curve.*

Proof: The results are immediate from the remarks about $\mathcal{E}$ and because $\mathcal{E}$ remains constant along a trajectory of solutions of the ODE. ♠

4.1 The period T

The integral for the period, with F as in equation (18),

$$T = 2\int_{\alpha_{\min}}^{\mathrm{invol}_\alpha(\alpha_{\min})} \frac{d\alpha}{\sqrt{2(F(\alpha_{\min}) - F(\alpha))}},$$

is integrated explicitly as follows. Define

$$a_s = \frac{-1+\sqrt{1+8(\frac{\alpha_{\min}}{\alpha_{eq}})^3}}{4(\frac{\alpha_{\min}}{\alpha_{eq}})^2}\,, \quad s_1 = \sqrt{1-8a_s^3}\,, \quad s_2 = 4a_s^3+s_1+1\,, \quad s_3 = \frac{2s_1}{s_2}\,.$$

Then

$$T = -\frac{4a_s^3\,\text{EllipticK}(s_3) - s_2\,\text{EllipticE}(s_3)}{\nu_0\ a_s\sqrt{s_2/6}}. \qquad (19)$$

The frequency is $\nu = 2\pi/T$. Taking $\epsilon \sim (\alpha_{eq} - \alpha_{\min})/\alpha_{eq}$ to zero yields the asymptotics (6), this time under limit L_4.

4.2 The ODE $u'' = 1/u^3 - 1$

On rescaling t and α to τ and u, respectively, the ODE (17) can be written $u'' = 1/u^3 - 1$. Whittaker [13] treats other applications of this ODE for $u(\tau)$.

- One application is that the ODE describes, in polar coordinates, central orbits with a constant attractive force to the origin. This problem, and the precession that is observed, was considered by Newton.
- A second application is that the ODE describes, again in polar coordinates, a particle sliding, with gravity acting, inside a smooth cone whose axis is vertical. Hooke, seeking analogues of central orbits showing precession, performed experiments on this arrangement.

5 Numerics for the initial value problem (1,9)

Plots of the orbits around the straight-line rolling equilibria are given by O'Reilly [7, Figure 4]. Figure 2 plots orbits about the equilibria of §2.

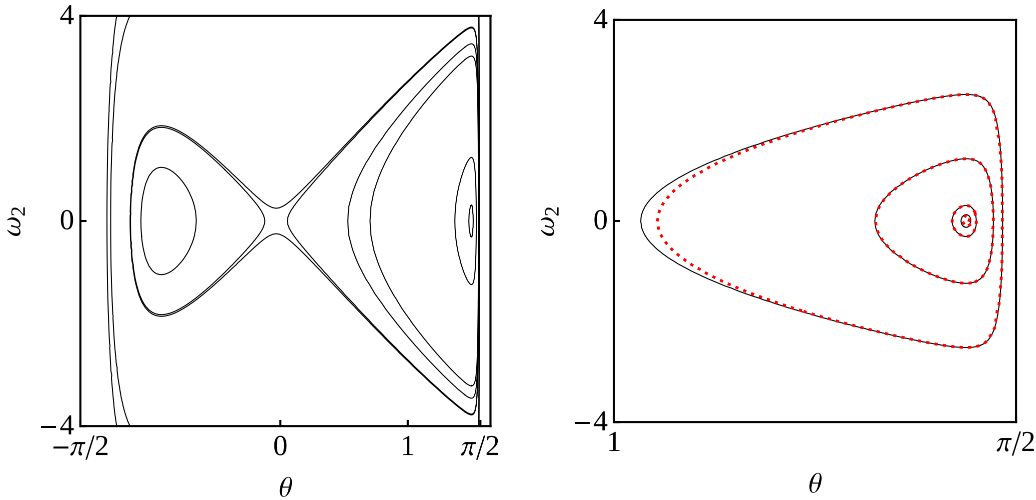

FIGURE 2: The solid curves are contours of the energy for the exact solutions perturbing about the $\omega_3 = 0$ family of equilibria in the case of a disk $\gamma = 2/3$, $\theta_{eq} = 1.5$. The right-hand plot shows, scaled up, a small portion close to the equilibrium: the dotted curves are the approximations of §4.

Equations (1) supplemented with $\dot{\phi} = -\cos(\theta)/\omega_1$ and further equations [4, 10] enable one to find other quantities related to the motions, the centroid, the point of contact with the plane, etc. Numerical integration showing these is available on the web.[1]

6 Further work?

It would be good to know if items in this study manifest themselves in experiments with Euler disks [9, 12, 2], for example in the sound measured. Another quantity which has been measured in experiments is the location of the centre. In experiments [2], oscillations about a small circular path of the centre are reported. The observations reported there are associated with a particular experiment with larger values of ω_3 than treated in §4.

Disks with finite thickness lead to very similar ODEs which are integrable [1, 8]. The approximations of this article should follow through for these.

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