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Piecewise linear approximation of nonlinear ordinary differential equations

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Abstract

The study of linear ordinary differential equations (ODEs) is an important component of the undergraduate engineering curriculum. However, most of the interesting behaviour of nature is described by nonlinear ODEs whose solutions are analytically intractable. We present a simple method based on the idea that the curve of the nonlinear terms of the dependent variable can be replaced by an approximate curve consisting of a set of line segments tangent to the original curve. This enables us to replace a nonlinear ODE with a finite set of linear inhomogeneous ODEs for which analytic solutions are possible. We apply this method to the cooling of a body under the combined effects of convection and radiation and demonstrate very accurate solutions with a relatively few number of line segments. Furthermore, we discuss how a number of key and usually disparate concepts of calculus are needed to apply this method, including continuity and differentiability, Taylor polynomials and optimisation.

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1 Introduction

Ordinary differential equations (ODEs) model phenomena in the world, ranging from wave motion to heat exchange; hence ODEs are extremely important to many scientists and engineers. They are solved using a variety of methods, some methods giving an exact solution use separation of variables or Laplace transforms. However, many ODEs involve non-linear terms and this makes solving them very difficult.

Modern day mathematical software such as MATLAB is able to graphically plot a solution to non-linear ODE. However, such a solution cannot be manipulated algebraically. Thus there is a need to develop new mathematical techniques to solve nonlinear ODEs, even if these techniques only provide approximate solutions. Perturbation techniques are one such mathematical method, providing approximate solutions to certain nonlinear ODEs. However, perturbation methods require the existence of a small parameter and yet many important nonlinear ODEs do not have a small parameter at all.

We introduce a new approach, one termed a *piece-wise linear emulating function method* [1]. The type of nonlinear ODE we consider is $L(y)+f(y)=0$, with given boundary conditions, where L is a linear differential operator (with constant coefficients) and $f(y)$ is some function of y.

2 Cooling of an object: exact solution

Heat and temperature are very closely related concepts, and they are often confused. The result of applying heat to an object is to raise its temperature. Similarly, as an object cools, its temperature drops as it loses heat to the surroundings. Here we consider the problem of combined convective-radiative cooling [2]:

$$mc\frac{dT}{dt} = -hS(T - T_a) - e\sigma S(T^4 - T_a^4), \tag{1}$$

where $T(t)$ is the temperature, m is the mass of the object being heated or cooled, T_a is the temperature of the air, S is the surface area of the object, h is the convective heat-transfer coefficient, and c is the *specific heat* of the object. We assume that h and c are independent of temperature. The factor e, called the *emissivity*, is a number between 0 and 1 that is characteristic of the material, and σ is the Stefan–Boltzmann constant.

The term on the left of Equation (1) is the rate of change of heat with time t. If $T > T_a$ then the surface is loosing heat, and if $T < T_a$ the surface is gaining heat.

The first term on the right is due to convective cooling. It models the transfer of heat by the mass motion of air (or a fluid) from one region of space to another. We assume forced convection, that is, there exists a column of slowly moving air across the surface of the cooling object (a slight breeze) which is the basis for Newton's Law of Cooling.

The second term is due to radiative cooling. Unlike convection which requires the presence of matter as a medium for the transfer of heat energy, radiation involves electromagnetic waves. The warmth we receive by a camp fire is mostly radiant energy, most of the air heated by the fire rises by convection and does not reach us. The rate at which an object at temperature T radiates energy is given by the Stefan–Boltzmann equation [3].

We wish to solve Equation (1) with the initial condition $T(0) = T_i$. To solve this equation we perform the following change of variables

$$T' = \frac{T}{T_i}\,, \quad T'_a = \frac{T_a}{T_i}\,, \quad t' = \frac{hSt}{mc}\,, \quad \epsilon = \frac{e\sigma T_i^3}{h}\,.$$

Making the above change of variables the differential equation becomes (upon dropping the dashes)

$$\frac{dT}{dt} + (T - T_a) + \epsilon(T^4 - T_a^4) = 0\,,$$

subject to the initial condition $T(0) = 1$. For the sake of exposition, we further assume that $T_a = 0$. The differential equation then simplifies to

$$\frac{dT}{dt} + T + \epsilon T^4 = 0\,. \tag{2}$$

This equation is a Bernoullie equation and can be solved exactly:

$$T^3 = \frac{e^{-3t}}{1 + \epsilon - \epsilon e^{-3t}}. \tag{3}$$

This solution has the correct asymptotic behaviour, that is, as $t \to \infty$, $T \to 0$.

3 Approximate solutions

We begin by writing Equation (2) as $T' + T + \epsilon f(T) = 0$ where $f(T) = T^4$. The aim is to use a linear piece-wise function to emulate $f(T)$ over the interval $(0, 1]$.

Figure 1 shows a plot of $f(T)$ with *two* linear functions, namely

$$f_l(T) = \begin{cases} 0, & 0 < T \leqslant T_0\,, \\ 4T - 3, & T_0 < T \leqslant 1\,, \end{cases}$$

where T_0 is a parameter that is determined by the intersection of the two straight lines. Here $T_0 = 0.75$. By replacing $f(T)$ with $f_l(T)$ we must solve

$$\frac{dT}{dt} + T + \epsilon f_l(T) = 0 \tag{4}$$

over each of the subintervals. This linear ODE can in each instance be solved by separation of variables or by using an integrating factor, the general solution is

$$T(t) = \begin{cases} \frac{3\epsilon}{1+4\epsilon} + C\,e^{-(1+4\epsilon)\,t}, & 0 < t \leqslant t_0\,, \\ A\,e^{-t}, & t > t_0\,. \end{cases}$$

Clearly we have the correct behaviour, as $t \to \infty$, that $T \to 0$. To solve the problem we must determine the constants C, A and t_0. We find C using the initial condition, $T(0) = 1$; ensuring continuity of the solutions at $T(t_0) = T_0$ determines both A and t_0. The steps involved are relatively straight forward and lead to

$$\begin{aligned} C &= \frac{1+\epsilon}{1+4\epsilon}\,, \\ t_0 &= \frac{1}{1+4\epsilon}\log\left[\frac{4(1+\epsilon)}{3}\right]\,, \\ A &= \frac{3}{4}\left[\frac{4(1+\epsilon)}{3}\right]^{1/(1+4\epsilon)}\,. \end{aligned}$$

We extend this approach of using a piece-wise linear function with three segments to emulate the nonlinear term of the heat equation, as shown in Figure 2. The three lines, defined over the subintervals $(0, T_0]$, $(T_0, T_2]$, $(T_2, 1]$

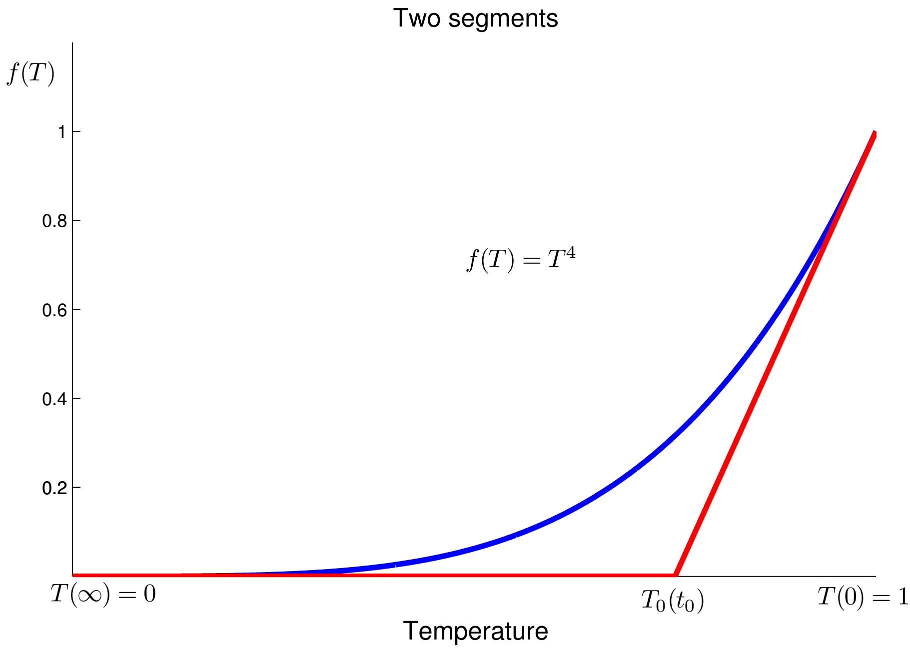

Figure 1: Emulating function with two segments.

with tangents to $f(T)$ at the points 0, T_1 and 1, respectively. The equations for the straight lines are found to be

$$f_l(T) = \begin{cases} 0, & 0 < T \leqslant T_0\,, \\ T_1^3(4T - 3T_1), & T_0 < T \leqslant T_2\,, \\ 4T - 3, & T_2 < T \leqslant 1\,. \end{cases}$$

The points T_0 and T_2 are points of intersection of two lines of consecutive subintervals. It is also clear from the Figure 2 that the locations of both T_0 and T_2 depend on T_1:

$$T_0 = \frac{3T_1}{4} \quad \text{and} \quad T_2 = \frac{3}{4}\left(\frac{1 - T_1^4}{1 - T_1^3}\right).$$

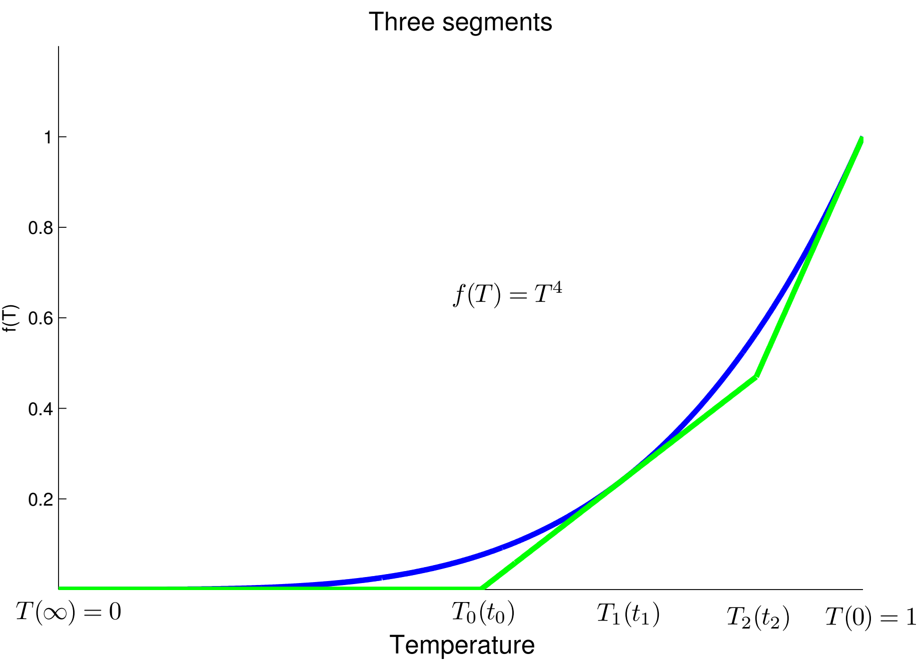

Figure 2: Emulating function with three segments.

Once T_1 is specified then both T_0 and T_2 can be determined.

To find the approximate solution we replace $f(T)$ with $f_l(T)$ and use the integrating factor method to find the solution to the equivalent *linear* first order, non-homogeneous ODE over each of the subintervals. The solutions are

$$T(t) = \begin{cases} \dfrac{3\epsilon}{1+4\epsilon} + C_2\, e^{-(1+4\epsilon)\,t}, & 0 \leqslant t < t_2\,, \\ \dfrac{3\epsilon T_1^4}{1+4\epsilon T_1^3} + C_1\, e^{-(1+4\epsilon T_1^3)\,t}, & t_2 \leqslant t < t_0\,, \\ A_0\, e^{-t}, & t \geqslant t_0\,, \end{cases}$$

where A_0, C_1, C_2, t_2 and t_0 are all constants that are to be determined. From the above, the solution for $T(t)$ displays the correct asymptotic behaviour.

The initial condition, $T(0) = 1$, helps determine

$$C_2 = \frac{1+\epsilon}{1+4\epsilon}\,,$$

which is exactly the same as for the case with two segments emulating the nonlinear term. The remaining four constants are found by requiring that $T(t)$ and its derivative are continuous at $T(t_2) = T_2$ and $T(t_0) = T_0$. Continuity of $T(t)$ and its derivative at t_2 gives rise to two transcendental equations which when solved give both

$$\begin{aligned} t_2 &= \frac{1}{1+4\epsilon} \log\left\{\frac{4}{3} \cdot \frac{(1+\epsilon)(1-T_1^3)}{[(1-T_1^4) + 4\epsilon T_1^3(1-T_1)]}\right\}, \\ C_1 &= \left(\frac{1+\epsilon}{1+4\epsilon T_1^3}\right) \left\{\frac{3}{4} \cdot \frac{\left[(1-T_1^4) + 4\epsilon T_1^3(1-T_1)\right]}{(1+\epsilon)(1-T_1^3)}\right\}^{4\epsilon(1-T_1^3)/(1+4\epsilon)}. \end{aligned}$$

Similarly, continuity of $T(t)$ and its derivative at t_0 leads to

$$t_0 = \frac{1}{1+4\epsilon T_1} \log\left[\frac{4C_1}{3} \cdot \frac{(1+4\epsilon T_1^3)}{T_1}\right],$$

$$A_0 \quad = \quad C_1(1+4\epsilon T_1^3)\left[\frac{3}{4C_1}\cdot\frac{T_1}{(1+4\epsilon T_1^3)}\right]^{4\epsilon T_1^3/(1+4\epsilon T_1^3)},$$

both in terms of C_1.

4 Comparison with exact solution

Since Equation (2) has the exact solution (3) it makes it possible to assess the accuracy for the emulating function method with both the two segments and three segments approximate solution, as well as the perturbation method (see Appendix A). But first we must get estimates for the single parameter ϵ. Recall that $\epsilon = e\sigma T_i^3/h$ where $\sigma = 5.67\times 10^{-8}\,\mathrm{W/(m^2K^4)}$. The emissivity e varies between 0 and 1 but for many surfaces it typically has a value of 0.8 [3]. Likewise, the convective heat transfer coefficient h, which is determined empirically, has a very large range of values, from 2 to $3500\,\mathrm{W/(m^2\,K)}$ (and even higher) [4]. Using these values estimates of ϵ range from 0.0004 to 0.6 at $T_i = 300\,\mathrm{K}$ and from 0.002 to 1.5 at $T_i = 400\,\mathrm{K}$. For comparison we consider two cases, $\epsilon = 0.1$ and $\epsilon = 1.5$.

Figures 3 and 4 show the difference between the exact and approximate solution for $\epsilon = 0.1$ and $\epsilon = 1.5$, respectively. For $\epsilon = 0.1$, the perturbation solution is excellent and both the two segment and three segment also provide very good solutions. Not surprisingly, the three segment solution is better than the two segment solution. It is possible to further improve the solution for the three segment case by the appropriate choice of T_1, as discussed below.

However, when $\epsilon = 1.5$, the perturbation solution is no longer the best solution. Although the difference between the exact and approximate solutions has increased, this increase is relatively modest. The relative error is less than 3%, quite an acceptable result considering that the two and three segment emulating function are a rather crude approximation to the nonlinear term.

In the three segment case, T_1 could be chosen at random, or a specific value

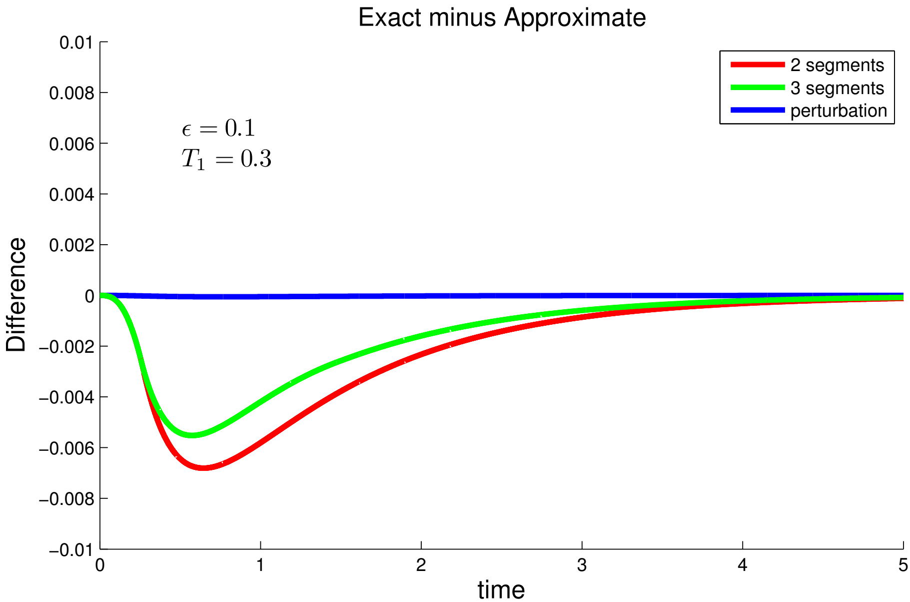

Figure 3: Comparison of exact solution with perturbation solution and the emulating function method with two and three segments for $\epsilon = 0.1$ and $T_1 = 0.3$.

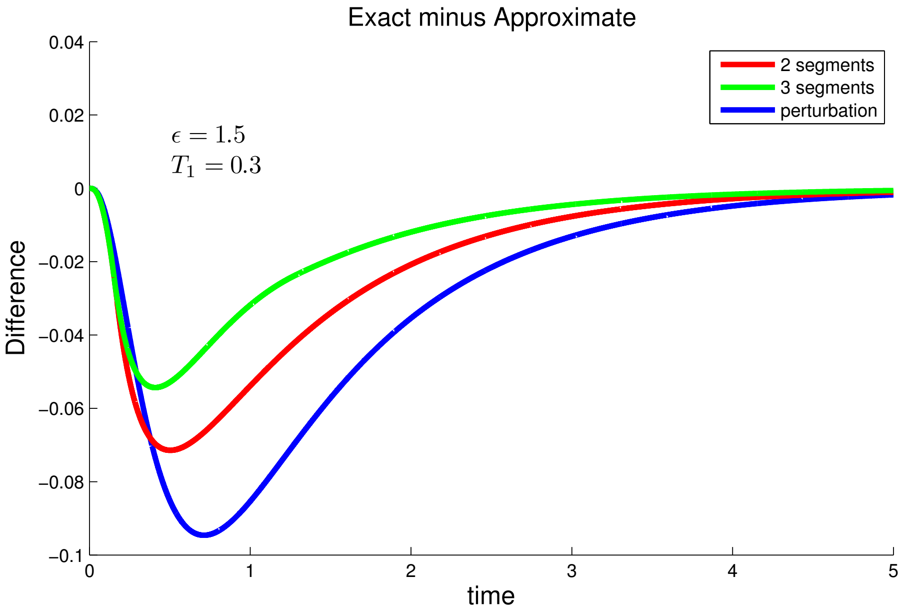

FIGURE 4: Comparison of exact solution with perturbation solution and the emulating function method with two and three segments for $\epsilon = 1.5$ and $T_1 = 0.3$.

could be used instead. However, it may be possible to choose T_1 that would optimise the solution. There are several ways that this could be done, one way is achieved by choosing T_1 to minimise $\int_0^1 |h(T;T_1)|\,dT$, where we define $h(T;T_1) = f(T) - f_l(T)$ as the difference function. This minimum is obtained by requiring

$$\frac{d}{dT_1}\int_0^1 |h(T;T_1)|\,dT = 0\,. \tag{5}$$

By breaking up the integral and integrating over each subinterval

$$\int_0^1 |h(T;T_1)|\,dT = \int_0^{T_0} T^4\,dT + \int_{T_0}^{T_2}(T^4 - 4T_1^3T + 3T_1^4)\,dT + \int_{T_2}^{1}(T^4 - 4T + 3)\,dT\,.$$

Using a generalised version of Leibniz's rule (see Appendix B) we get

$$\frac{d}{dT_1}\int_0^1 |h(T;T_1)|\,dT = 2T_1^5 - 2T_1^4 - 5T_1^2 + 8T_1 - 3\,. \tag{6}$$

This fifth order polynomial has one zero in the interval $0 < T_1 < 1$, namely $T_1 \approx 0.691$. Figures 5 and 6 show that this value for T_1 significantly improves accuracy.

5 Conclusion

The emulating function method, whereby a nonlinear term in an ODE is approximated by a piecewise linear function consisting of two, three or more line segments, provides a conceptually simple avenue to solving nonlinear ODEs. We demonstrated, in the case of an object cooling under the combined effects of convection and radiation, the emulating function method provides accurate algebraic solutions even with as few as three line segments. This is also true even in the case when there is no small parameter ϵ.

The method outlined here also has pedagogical value. Students employ several key concepts from elementary calculus: continuity and differentiability, (first

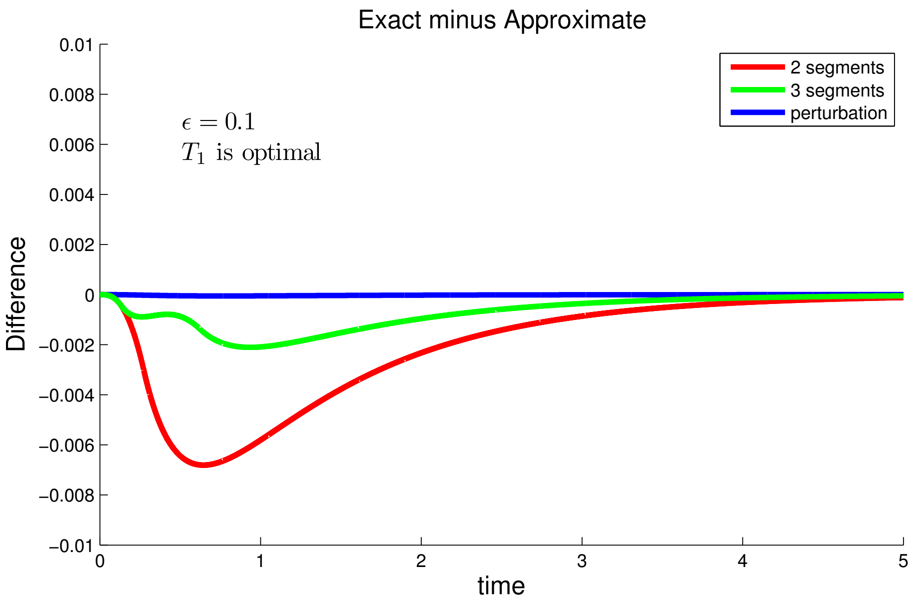

Figure 5: Comparison of exact solution with perturbation solution and the emulating function method with two and three segments for $\epsilon = 0.1$ and T_1 is optimised.

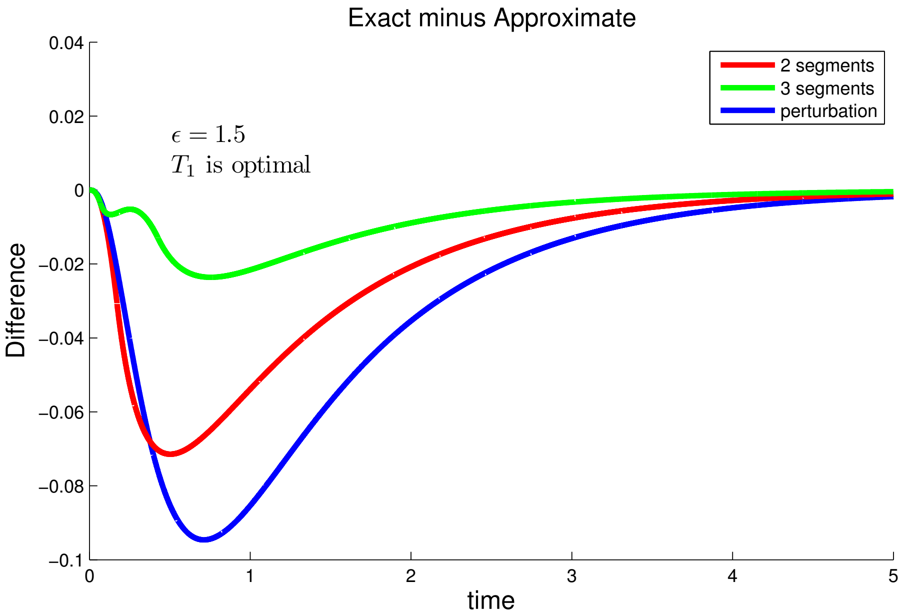

FIGURE 6: Comparison of exact solution with perturbation solution and the emulating function method with two and three segments for $\epsilon = 0.1$ and T_1 is optimised.

order) Taylor polynomials, and optimisation (among others). These concepts appear as disparate ideas in the traditional undergraduate course in applied mathematics or engineering and are often seen as mere methods to problems lacking significance in the real word. We have set semester long projects based on the emulating function method to solve a variety of first order nonlinear ODEs and the experience of our students has been a very positive one.

Finally, piecewise linear functions do have important applications. They are part of a vast body of research in *Approximation Theory*. Simple applications in the undergraduate degree arise in the study of Fourier analysis of a triangular or sawtooth wave function or in the study of linear ODEs with forcing terms that are piecewise linear ramp functions. Not so trivial problems arise in Operations Research where the nonlinear boundary of a feasible solution space is approximated with a set of piecewise linear constraints.

A Perturbation solution

Assuming that ϵ is small, we formally write the solution as

$$T(t) = T^{(0)}(t) + \epsilon T^{(1)}(t) + \epsilon^2 T^{(2)}(t) + \cdots .$$

Substituting this expression into Equation (2) and applying standard regular perturbation thoery gives the approximate solution

$$T(t) = e^{-t} + \frac{\epsilon}{3}(e^{-4t} - e^{-t}) + \frac{\epsilon^2}{9}(2e^{-7t} - 4e^{-4t} + 2e^{-t}) + \mathcal{O}(\epsilon^3).$$

B Leibniz's rule

Using Leibniz's rule

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}\,,$$

where α is a parameter and the limits of integration a and b may be functions of α.

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