A remark on energy optimal strategies for a train movement

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Abstract

This article introduces the notion of the critical time in the problem of the energy efficient train control and its calculation in some particular cases. We apply some results of non-linear parametric optimization to show that the number of optimal control levels depends on the relation between the given time of the journey and this critical time. Furthermore, we derive equations for the computation of the switching times. I emphasise exact forms of solutions with a minimal use of numerical mathematics. The results can be used to find the values of the switching times only by solving algebraic equations and to analyse the behaviour of the results with respect to given entry parameters of the problem.

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1 Introduction

The basic problem of the energy efficient train control was formulated and solved in some particular cases by Horn [5] in 1971. That was not long after the publication of the general form of the Pontryagin principle and the related mathematical tools. Since then, it has become a typical problem that can be solved with use of these means.

Many articles discussing this topic have appeared especially during the nineties. The type of the optimal strategy consisting of four successive control levels (full power, speed holding, coasting and full braking) was presented by Howlett and Cheng [6, 9]. Among articles dealing with various modifications of the basic problem we mention, for example, that Pudney and Howlett [16] considered a vehicle with discrete control settings and speed limits. Howlett and Cheng [7] discussed a track with a non-zero gradient. Both of these assumptions were considered by Cheng et al. [3] and Khmelnitsky [12]. Howlett and Pudney [11] summarised these results. These articles formed the theoretical background for the introduction of systems for calculating efficient driving advice during a journey. The Metromiser and later FreightMiser were developed as on-board systems which displayed efficient driving advice to the driver and were used with positive results to timetabled suburban and long-haul trains. Note that some alternative approaches to this and related problems were discussed and solved by Han et al. [4], Howlett and Leizarowitz [8, 10], Li et al. [13], Liu and Golovitcher [14] and Pickhardt [15].

This article discusses a slightly different approach to the basic problem of the energy efficient train control which is based on previous results and extends them in a specific way. The formulation of the problem is described in Section 2. The previous articles lead us to two possible variants of optimal strategy (either with the speed holding phase or without it). Section 3 recalls and extends the application of the Pontryagin principle and related tools to the given problem of energy efficient train control. The main purpose of this article is to determine which of the two possible strategies occurs for given values of entry parameters. Therefore, Section 5 introduces the notion of the critical time to describe the behaviour of the problem with respect to some given time of the journey. We use the theoretical background of nonlinear parametric programming from Section 4 to find an equation for its calculation and, furthermore, we derive relations which enable us to compute the switching times between the individual phases of control for the most common cases of linear and quadratic resistance function. We do not present the way of explicit calculation of the switching times just to determine the relevant values, but their specification is needed to derive the value of the critical time. Section 6 summarizes the method of computation.

2 Formulation of the problem

Throughout this article we study the problem of the energy efficient train control in the following form. We wish to minimize the objective functional

$$J = \int_0^T u^+(t)\, x_2(t)\, dt \tag{1}$$

with respect to the system of differential equations

$$\dot{x}_1 = x_2\,, \tag{2}$$
$$\dot{x}_2 = u(t) - r(x_2)\,, \tag{3}$$

and boundary conditions

$$x_1(0) = 0\,, \quad x_2(0) = 0\,, \tag{4}$$
$$x_1(T) = L\,, \quad x_2(T) = 0\,. \tag{5}$$

The function u^+ is defined as

$$u^+(t) := \begin{cases} u(t)\,, & \text{for } u(t) > 0\,, \\ 0\,, & \text{for } u(t) \le 0\,. \end{cases}$$

We assume that the control u is a piecewise continuous function mapping $[0, T]$ into $[-\alpha, \beta]$, where $\alpha, \beta > 0$ are given constants, and $r = r(x_2)$ is a differentiable function (with respect to x_2) with the properties $r, r' \ge 0$ and $r'(x_2)\, x_2$ is nondecreasing for $x_2 \ge 0$. Sections 3, 5 and 6 illustrate our considerations utilizing linear and quadratic functions r (fulfilling the above mentioned properties).

We recall that the problem (1)–(5) describes the motion of a train along a straight level track of length $L > 0$ with minimal consumption of electric energy J. We assume that the mass of the train $m = 1$. Phase coordinates x_1 and x_2 correspond to position and speed of the train. The given parameter T represents the time that is available according to the timetable for the train to complete the track. The function r represents the frictional resistance.

We do not present physical units as they are not essential from the mathematical point of view.

3 Application of Pontryagin principle and related tools

First, we must enumerate the minimum time $T_{\min}$ that it is possible to complete the track within. Using the standard procedure we arrive at the 'bang-bang' control solving the corresponding minimum time problem. Then we set up an equation for $T_{\min}$. In particular, if $r(x_2) = kx_2$ $(k > 0)$ then the time

$$T_{\min} = \frac{1}{k} \ln \eta \,, \tag{6}$$

where $\eta > 1$ has to satisfy the relation

$$(\alpha + \beta)\, e^{Lk^2/(\alpha+\beta)} \eta^{\alpha/(\alpha+\beta)} - \alpha\eta - \beta = 0 \,.$$

Similarly, the value of $T_{\min}$ is determined for quadratic resistance function $\left(r(x_2) = k(x_2)^2, k > 0\right)$ from equation

$$T_{\min} = t^* + \frac{1}{\sqrt{k\alpha}} \arctan\left[\sqrt{\frac{\beta}{\alpha}} \tanh\left(\sqrt{k\beta} t^*\right)\right] , \tag{7}$$

where t^* is calculated from equation

$$\alpha \cosh^2\left(\sqrt{k\beta} t^*\right) + \beta \sinh^2\left(\sqrt{k\beta} t^*\right) = \alpha e^{2kL} \,. \tag{8}$$

Throughout this article we assume that the given time T satisfies the relation $T > T_{\min}$.

Now recall the assertion which yields the solution of the energy efficient control problem (1)–(5) [6].

Theorem 1 *Let* $(\hat{x}_1(t), \hat{x}_2(t); \hat{u}(t))$, $t \in \langle 0, T\rangle$ *be the energy optimal solution of (1)–(5). Then there exist* t_1, t_2, t_3, *where* $0 < t_1 \le t_2 < t_3 < T$, *such that*

$$\hat{u}(t) = \begin{cases} \beta\,, & \textit{for } 0 \le t < t_1\,, \\ r(\hat{x}_2(t)) = \textit{constant}\,, & \textit{for } t_1 < t < t_2\,, \\ 0\,, & \textit{for } t_2 < t < t_3\,, \\ -\alpha\,, & \textit{for } t_3 < t \le T\,. \end{cases} \tag{9}$$

Our next task is the calculation of switching times t_1, t_2, t_3 with respect to the relation between t_1 and t_2. *The type of this relation cannot be specified directly from Pontryagin principle.*

Let us assume that $t_1 = t_2$. Then we can easily calculate the values of the switching times by integration of Equations (2) and (3) on separate intervals, comparing values of position and speed in boundary points of these time intervals (that is, in $t = t_1 = t_2$ and $t = t_3$) and involving conditions (4) and (5). Of course, the second phase (speed-holding) is omitted in this consideration. In particular, let us consider the linear resistance function r. We obtain an equation for the unknown t_3:

$$Lk^2 + \alpha k T - \alpha k t_3 = \beta \ln\left(\frac{\alpha}{\beta} e^{kT} - \frac{\alpha}{\beta} e^{kt_3} + 1\right)\,. \tag{10}$$

Consequently, the value of time $t_1 = t_2$ is determined from

$$t_1 = \frac{1}{k} \ln\left(\frac{\alpha}{\beta} e^{kT} - \frac{\alpha}{\beta} e^{kt_3} + 1\right)\,. \tag{11}$$

In the case of quadratic resistance we obtain similarly the equation for calculation of time t_3:

$$\sqrt{\frac{k}{\beta}}\, \operatorname{arcsinh}\left\{\sqrt{\frac{\alpha}{\beta}} e^{kL} \left|\sin\left[\sqrt{\alpha k}\,(T - t_3)\right]\right|\right\}$$

$$
\begin{aligned}
= \quad & \sqrt{\frac{k}{\beta}} \coth \operatorname{arcsinh} \left\{ \sqrt{\frac{\alpha}{\beta}} e^{kL} \left| \sin \left[\sqrt{\alpha k} \left(T - t_3 \right) \right] \right| \right\} \\
& - \sqrt{\frac{k}{\alpha}} \cot \left[\sqrt{\alpha k} \left(T - t_3 \right) \right] + k t_3 \,. \qquad (12)
\end{aligned}
$$

Afterwards, we compute the value of time t_1 from

$$
t_1 = \frac{1}{\sqrt{\beta k}} \operatorname{arcsinh} \left\{ \sqrt{\frac{\alpha}{\beta}} e^{kL} \left| \sin \left[\sqrt{\alpha k} \left(T - t_3 \right) \right] \right| \right\} . \qquad (13)
$$

The case $t_1 < t_2$ is a bit more complicated. We must determine the values of three unknown variables t_1, t_2 and t_3. However, position and speed with boundary conditions represent only two equations. Therefore, it is necessary to consider the Hamilton function,

$$
H = -u^{+} x_2 + \lambda_1 x_2 + \lambda_2 \left(u - r \left(x_2 \right) \right) ,
$$

and under various suitable choices of the independent variable t we utilize the property $H =$ constant on $[0, T]$. To illustrate this we consider the linear resistance function r and obtain

$$
\begin{aligned}
H(0) &= \lambda_2(0)\,\beta \,, \\
H(t_1^-) &= -\beta x_{2\max} + C_1 x_{2\max} + \lambda_2(t_1^-)\left(\beta - k x_{2\max}\right) \,, \\
H(t_1^+) &= -k\left(x_{2\max}\right)^2 + C_1 x_{2\max} \,, \\
H(t_2^+) &= C_1 x_{2\max} - \lambda_2(t_2^+)\, k x_{2\max} \,, \\
H(t_3^-) &= C_1 x_2(t_3) \,, \\
H(T) &= -\lambda_2(T)\,\alpha \,,
\end{aligned}
$$

where $x_{2\max}$ denotes the highest speed that the train reaches along its track (on interval $[t_1, t_2]$) and $H(t_1^-)$ (respectively $H(t_1^+)$) denotes the corresponding one-sided limit (similarly in the remaining cases).

Lagrange multiplicators λ_1, λ_2 are solutions of the adjoint system

$$
\dot{\lambda}_1 = 0, \qquad (14)
$$

$$\dot{\lambda}_2 \;=\; \hat{u}^+ - \lambda_1 + k\lambda_2\,. \tag{15}$$

Recall that the variable λ_2 must be continuous on $[0, T]$. Thus, it holds that $\lambda_2(t_1^-) = \lambda_2(t_1^+) = \lambda_2(t_1)$ and analogously in other cases. Further, the relation $\lambda_2(t) = x_{2\mathrm{max}}$ must be fulfilled on (t_1, t_2) (this follows directly from Pontryagin principle for this type of optimal regulation). Hence, λ_2 is constant here and therefore $\dot{\lambda}_2(t) = 0$ for $t \in (t_1, t_2)$. Consequently, relations (14) and (15) imply that

$$\lambda_1(t) \equiv C_1 = 2kx_{2\mathrm{max}}\,. \tag{16}$$

Now, we use the relation for Hamilton function in t_1^+ to derive

$$H(t) \equiv k\left(x_{2\mathrm{max}}\right)^2$$

for $t \in [0, T]$. The value of $H(t_3)$ and Equation (16) lead us to conclude that

$$x_2(t_3) = \frac{x_{2\mathrm{max}}}{2}\,. \tag{17}$$

This equation represents the required third equation that is necessary to derive an equation for calculation of the switching times in this case. Thus, for linear resistance it is possible to derive (analogously to the case $t_1 = t_2$ with use of Equation (17)) the following equation for t_2

$$\begin{aligned} &Lk^2 + \alpha kT + \alpha kt_2 - \alpha\ln 2 - \alpha kt_2 e^{k(T-t_2)} \\ &= \left[\alpha e^{k(T-t_2)} - 2\alpha - \beta\right]\ln\left[-\frac{\alpha}{\beta}e^{k(T-t_2)} + \frac{2\alpha}{\beta} + 1\right]. \end{aligned} \tag{18}$$

The remaining switching times t_1 and t_3 then are

$$t_1 \;=\; -\frac{1}{k}\ln\left[-\frac{\alpha}{\beta}e^{k(T-t_2)} + \frac{2\alpha}{\beta} + 1\right], \tag{19}$$

$$t_3 \;=\; t_2 + \frac{1}{k}\ln 2\,. \tag{20}$$

Analogously as in the previous case we determine the values of t_1, t_2 and t_3 for quadratic resistance function. The value t_1 is calculated from equation

$$\begin{aligned} & -\sqrt{\frac{\beta}{\alpha}} \tanh\left(\sqrt{\beta k} t_1\right) \arctan\left[\sqrt{\frac{\beta}{\alpha}} \frac{2}{3} \tanh\left(\sqrt{\beta k} t_1\right)\right] \\ = \; & \ln\left\{\frac{2}{3} \cos \arctan\left[\sqrt{\frac{\beta}{\alpha}} \frac{2}{3} \tanh\left(\sqrt{\beta k} t_1\right)\right]\right\} \\ & - \ln\cosh\left(\sqrt{\beta k} t_1\right) + \frac{1}{2} + kL - (T - t_1)\sqrt{\beta k} \tanh\left(\sqrt{\beta k} t_1\right) \end{aligned} \tag{21}$$

and consequently the values of switching times t_2 and t_3 are determined by

$$t_3 = T - \frac{1}{\sqrt{\alpha k}} \arctan\left[\sqrt{\frac{\beta}{\alpha}} \frac{2}{3} \tanh\left(\sqrt{\beta k} t_1\right)\right], \tag{22}$$

$$t_2 = t_3 - \frac{1}{2\sqrt{\beta k} \tanh\left(\sqrt{\beta k} t_1\right)}. \tag{23}$$

4 A nonlinear parametric optimization problem

This section describes a part of the mathematical theory of nonlinear parametric optimization which is relevant in our further investigations. Bank [1] gives precise proofs of the following theorems and other useful results.

In this section we assume the following nonlinear parametric optimization problem:

$$\min\{f(\mathbf{x}, \lambda) \mid \mathbf{x} \in M(\lambda)\}, \quad \lambda \in \Lambda, \tag{24}$$

where $M(\lambda) \subset X$, X and Λ are metric spaces and f is a function mapping $X \times \Lambda$ into $\mathbf{R} \cup \{+\infty, -\infty\}$. Further, let us denote

$$\varphi : \lambda \to \varphi(\lambda) := \inf_{\mathbf{x} \in M(\lambda)} f(\mathbf{x}, \lambda)$$

the function describing the optimal value of the cost function f from problem (24) in dependence on the vector of parameters λ. Further, let

$$\psi : \lambda \to \psi(\lambda) := \{\mathbf{x} \in M(\lambda) \mid f(\mathbf{x}, \lambda) = \varphi(\lambda)\}$$

denote a mapping which assigns to each vector of parameters λ a set of all optimal solutions $\mathbf{x} \in X$ of problem (24).

Definition 2 *Let (X, d_X) and (Λ, d_Λ) be metric spaces. Point-to-set mapping $\Gamma : \Lambda \to 2^X$ is a function mapping each $\lambda \in \Lambda$ into a (possibly empty) subset $\Gamma(\lambda)$ of X.*

Remark 3 As customary, for a subset A of the metric space X and for arbitrary $\epsilon > 0$ the ϵ-*neighbourhood* of the set A is the set

$$U_\epsilon A := \{\mathbf{x} \in X \mid d_X(\mathbf{x}, A) < \epsilon\}, \quad \text{where} \quad d_X(\mathbf{x}, A) = \inf_{\mathbf{y} \in A} d_X(\mathbf{x}, \mathbf{y})$$

and d_X denotes the corresponding metric. If A is an empty set, then $d_X(\mathbf{x}, A)$ is by definition equal to $+\infty$. To avoid misunderstanding, we denote by the symbol $V_\epsilon B$ the ϵ-neighbourhood of the set $B \subset \Lambda$. We further assume the euclidean metric.

Definition 4 *A point-to-set mapping $\Gamma : \Lambda \to 2^X$ is said to be*

1. *closed at a point λ^0 if for each pair of sequences $\{\lambda^t\} \subset \Lambda$ and $\{\mathbf{x}^t\} \subset X$, $t = 1, 2, \ldots$ with the properties*

$$\lambda^t \to \lambda^0, \quad \mathbf{x}^t \in \Gamma\lambda^t, \quad \mathbf{x}^t \to \mathbf{x}^0$$

   it follows that $\mathbf{x}^0 \in \Gamma\lambda^0$;

2. *upper semicontinuous (according to Berge or, simply, B) at a point λ^0, if for each open set Ω containing $\Gamma\lambda^0$ there exists a $\delta = \delta(\Omega) > 0$ such that $\Gamma\lambda \subset \Omega$ for every $\lambda \in V_\delta\{\lambda^0\}$;*

3. *lower semicontinuous (according to Berge or, simply, B) at a point* λ^0, *if for each open set* Ω *satisfying* $\Omega \cap \Gamma\lambda^0 \neq \emptyset$ *there exists a* $\delta = \delta(\Omega) > 0$ *such that* $\Omega \cap \Gamma\lambda \neq \emptyset$ *for every* $\lambda \in V_\delta\{\lambda^0\}$;
4. *upper semicontinuous (according to Hausdorff or, simply, H) at a point* λ^0, *if for each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\Gamma\lambda \subset U_\epsilon \Gamma\lambda^0$ *for every* $\lambda \in V_\delta\{\lambda^0\}$;
5. *lower semicontinuous (according to Hausdorff or, simply, H) at a point* λ^0, *if for each* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $\Gamma\lambda^0 \subset U_\epsilon \Gamma\lambda$ *for every* $\lambda \in V_\delta\{\lambda^0\}$.

Remark 5 We use, according to Bank [1], the following abbreviations: *u.s.c.-B* for upper semicontinuous (B) mapping, *l.s.c.-B* for lower semicontinuous (B) mapping and analogically *u.s.c.-H*, *l.s.c.-H*.

Remark 6 The following implications hold [1]:

$$\text{u.s.c.-B} \Rightarrow \text{u.s.c.-H}\,, \quad \text{l.s.c.-H} \Rightarrow \text{l.s.c.-B}\,.$$

Definition 7 *A point-to-set mapping* $\Gamma : \Lambda \to 2^X$ *is continuous at* λ^0 *if it is u.s.c.-H and l.s.c.-B at* λ^0.

Lemma 8 *If the mapping* Γ *is u.s.c.-H at* λ^0 *and if the set* $\Gamma\lambda^0$ *is closed, then the mapping* Γ *is closed at* λ^0.

Now, let us assume the problem (24) again. The following theorems describe the properties of continuity of the mappings which determine the optimal solution of the problem.

Theorem 9 *Let M be closed at λ^0, $M\left(\lambda^0\right)$ be non-empty, f be continuous and the metric space X be compact. Then φ is lower semicontinuous at λ^0; φ is also upper semicontinuous at λ^0 if and only if the mapping ψ is u.s.c.-B at λ^0.*

Theorem 10 *φ is upper semicontinuous at λ^0 if M is l.s.c.-B at λ^0 and f is upper semicontinuous on $M\left(\lambda^0\right) \times \left\{\lambda^0\right\}$.*

5 Introduction and calculation of the critical time

Section 3 derived the values of switching times for both optimal strategies, that is, including the speed holding phase or not. However, there still remains the question which of these two strategies is optimal for given entry parameters of the problem. One way of answering this question consists in the calculation of t_1, t_2 and t_3 for both cases. Then comparing the obtained values of the cost functional we arrive at the optimal choice.

To illustrate this idea, we specify J by use of expression (1) in all cases without the phase of speed holding as

$$J = \beta \hat{x}_1\left(t_1\right) = -\frac{\beta^2}{k^2} + \frac{\beta^2}{k} t_1 + \frac{\beta^2}{k^2} e^{-kt_1}, \tag{25}$$

for linear resistance function, and

$$J = \frac{\beta}{k} \ln\cosh\left(\sqrt{\beta k}\, t_1\right), \tag{26}$$

for quadratic resistance function. For an optimal strategy containing the speed holding phase is the value, for linear resistance,

$$J = \beta \hat{x}_1\left(t_1\right) + \int_{t_1}^{t_2} k\left(\hat{x}_2\right)^2(t)\, dt = -\frac{\beta^2}{k^2} + \frac{\beta^2}{k} t_1 + \frac{\beta^2}{k^2} e^{-kt_1} + k\left(x_{2\max}\right)^2\left(t_2 - t_1\right) \tag{27}$$

whereas for quadratic resistance

$$J = \beta \hat{x}_1(t_1) + \int_{t_1}^{t_2} k\,(\hat{x}_2(t))^3\,dt = \frac{\beta}{k} \ln\cosh\left(\sqrt{\beta k}\,t_1\right) + k\,(x_{2\max})^3\,(t_2 - t_1)\,. \tag{28}$$

Numerical calculations show that for fixed values of entry parameters α, β, k and L the choice of the optimal strategy depends only on the given value of T. In particular, we conjecture that there exists a certain critical value of T, denoted as T_{cr}, such that for T satisfying the condition $T_{min} < T < T_{cr}$ the optimal strategy fulfils $t_1 = t_2$ and for $T > T_{cr}$ the inequality $t_1 < t_2$ holds. In the sequel we confirm this conjecture and derive an equation for calculation of T_{cr}.

We describe the behaviour of our optimization problem with respect to the given parameter T. Therefore, it is convenient to use the mathematical theory of parametric programming. To simplify the analysis, we assume that there exists a value T_{max}, sufficiently large, with the property $T_{min} \le T \le T_{max}$ and consider only the case of the linear resistance (for quadratic resistance there is presented only the equation that can be used for computation of T_{cr}).

Using Theorem 1 we rewrite the problem (1)–(5) into the following non-linear programming problem. We wish to minimize the objective function

$$J = \frac{\beta^2}{k}(t_2 - t_1)\left(1 - e^{-kt_1}\right)^2 + \frac{\beta^2}{k^2}\left(kt_1 + e^{-kt_1} - 1\right) \tag{29}$$

with respect to equations

$$\begin{aligned} \alpha\left[e^{k(T-t_3)} - 1\right] - \beta\left(1 - e^{-kt_1}\right)e^{k(t_2-t_3)} &= 0\,, & (30)\\ \alpha\,(t_3 - T) - kL + \beta\left(t_2 - t_2 e^{-kt_1} + t_1 e^{-kt_1}\right) &= 0\,, & (31) \end{aligned}$$

and inequalities

$$0 \le t_1 \le t_2 \le t_3 \le T\,. \tag{32}$$

The constraints (30) and (31) can be derived by use of the boundary conditions (4) and (5). Since the set of all admissible solutions has to be closed, we consider the inequalities for t_1, t_2 and t_3 in the form (32) (the cases $0 = t_1$, $t_2 = t_3$ and $t_3 = T$ cannot be optimal provided $T > T_{\min} > 0$).

Now we investigate the continuous dependence of the solution (t_1, t_2, t_3) of the problem (29)–(32) on parameter T. Therefore, we introduce the following assumption:

Hypothesis 11 *The point-to-set mapping $M(T)$ is continuous in T for all $T \geq T_{min}$.*

Here $M(T)$ denotes the set of all admissible solutions of the given problem, that is, the set of all (t_1, t_2, t_3) satisfying (30)–(32) for a given T.

Note that the validity of Hypothesis 11 can be verified under the specified values k, α, β and L.

Lemma 12 *Let the Hypothesis 11 be fulfilled. Then the point-to-set mapping*

$$\psi(T) := \{(t_1, t_2, t_3) \in M(T) \mid J(t_1, t_2, t_3; T) = \varphi(T)\} ,$$

where

$$\varphi(T) := \inf_{(t_1, t_2, t_3) \in M(T)} J(t_1, t_2, t_3; T) ,$$

is u.s.c.-B for every $T_{min} \leq T \leq T_{max}$.

Proof: We apply Theorem 9 and Theorem 10 to our problem. The mapping $\varphi(T)$ is represented now by the optimal value of the cost functional J which is given by Equation (29) for a fixed value of T. The mapping $\psi(T)$ is a point to set mapping which to every fixed value of $T \geq T_{\min}$ assigns a set of all optimal solutions of the given nonlinear programming problem, that is, a set of all optimal values (t_1, t_2, t_3). Under Hypothesis 11 the mapping $M(T)$ is also l.s.c.-B for every $T \geq T_{\min}$. Moreover, $J = J(t_1, t_2, t_3; T)$ from relation (29) is upper semicontinuous on $\mathbb{R}^3 \times \mathbb{R}$ (it is continuous). Then, $\varphi(T)$ is an upper semicontinuous mapping for every $T \geq T_{\min}$ according to Theorem 10. Further, we note that $M(T)$ is a non-empty set for every $T \geq T_{\min}$. Metric space X occurring in Theorem 9 is in our case the set of all (t_1, t_2, t_3) satisfying inequalities (32) and thus X is compact because of $T \leq T_{\max}$. Further, it is necessary for the mapping $M(T)$ to be closed in T for every $T \geq T_{\min}$. This property follows from Lemma 8 since $M(T)$ is, according to Hypothesis 11, u.s.c.-H at T and the set of all (t_1, t_2, t_3) satisfying (30), (31) and (32) is closed. Therefore, by Theorem 9 the mapping $\psi(T)$ is u.s.c.-B at T for every $T_{\min} \leq T \leq T_{\max}$. ♠

The assertion of Lemma 12 ensures that if we choose some fixed T^* and the corresponding optimal solution (t_1^*, t_2^*, t_3^*) of (29)–(32), then considering T sufficiently close to T^* we obtain a solution (t_1, t_2, t_3) close to (t_1^*, t_2^*, t_3^*).

Now we introduce the notion of the critical time T_{cr} and present its computation.

Definition 13 *A parameter T is said to be the* critical time *of the problem (29)–(32), denote it as T_{cr}, if there exists an $\epsilon > 0$ such that for $T = T_{cr}$ the nonlinear programming problem (29)–(32) has an optimal solution with property $t_1 = t_2$ and for $T \in (T_{cr}, T_{cr} + \epsilon)$ the corresponding optimal solution satisfies $t_1 < t_2$.*

Lemma 14 *Let T_{cr} be the critical time of the problem (29)–(32) and let the Hypothesis 11 be fulfilled. Then T_{cr} is the unique positive solution of*

$$\alpha k T_{cr} - \alpha \ln 2 + L k^2 + (\alpha + \beta) \ln \left(\frac{2\alpha + \beta}{\beta + \alpha e^{k T_{cr}}} \right) = 0 . \qquad (33)$$

Proof: Section 3 found the values of t_1, t_2 and t_3 under assumption $t_1 < t_2$. Due to Lemma 12, $\psi(T)$ is u.s.c.-B for every $T \geq T_{\min}$. Hence, letting

$t_2 \to t_1^+$ and comparing both calculations performed for $t_1 < t_2$ and $t_1 = t_2$ we arrive at the determination of the relation for T_{cr}.

More precisely, Equation (19) determines time t_1 provided $t_1 < t_2$. Now we use the relation $t_1 = t_2$ to obtain

$$t_2 = -\frac{1}{k} \ln \left[-\frac{\alpha}{\beta} e^{k(T_{cr} - t_2)} + \frac{2\alpha}{\beta} + 1 \right].$$

This relation leads us to the expression

$$t_2 = -\frac{1}{k} \ln \left[\frac{2\alpha + \beta}{\beta + \alpha \exp(kT_{cr})} \right].$$

This value is substituted to Equation (18) and after some adjustments we obtain Equation (33).

To show that Equation (33) admits a unique solution we put

$$F(T) := \alpha k T - \alpha \ln 2 + Lk^2 + (\alpha + \beta) \ln \left(\frac{2\alpha + \beta}{\beta + \alpha e^{kT}} \right)$$

denoting the function which describes the left-hand side of Equation (33). Then

$$F\left(\frac{\ln 2}{k} \right) = Lk^2 > 0 .$$

Further,

$$\begin{aligned} \lim_{T\to\infty} F(T) &= \lim_{T\to\infty} \left[\alpha k T - (\alpha + \beta) \ln \left(\beta + \alpha e^{kT} \right) \right] - \alpha \ln 2 + Lk^2 \\ &\quad + (\alpha + \beta) \ln (2\alpha + \beta) = -\alpha \ln 2 + (\alpha + \beta) \ln (2\alpha + \beta) \\ &\quad + Lk^2 + \lim_{T\to\infty} [\alpha k T - (\alpha + \beta)(\ln \alpha + kT)] = -\infty , \end{aligned}$$

because $\ln\left(\beta + \alpha e^{kT}\right) \approx \ln\left(\alpha e^{kT}\right) = \ln \alpha + kT$ as $T \to \infty$. Moreover,

$$F'(T) = \alpha k - (\alpha + \beta) \frac{\alpha k e^{kT}}{\beta + \alpha e^{kT}} = \alpha k \left[1 - \frac{(\alpha + \beta) e^{kT_{cr}}}{\alpha e^{kT} + \beta} \right] < 0$$

for $T > 0$ and this shows the uniqueness of the positive solution of (33). ♠

An easy consideration shows that if for some fixed $T = T^*$ the optimal solution of (29)–(32) has the property $t_1 < t_2$, then for every $T \geq T^*$ the corresponding optimal solution of (29)–(32) has the same property. In other words, if the optimal trajectory contains the speed holding phase for some $T = T^*$, then the speed holding phase will be contained in every optimal strategy with $T > T^*$. The proof of this claim is performed in an analogous way to the case of the proof of Lemma 14. Indeed, assume that there exists a parameter T^{**} such that for $T \in (T^{**} - \epsilon, T^{**})$, $\epsilon > 0$ being small enough, the problem (29)–(32) has an optimal solution with property $t_1 < t_2$ and for $T = T^{**}$ the corresponding optimal solution satisfies $t_1 = t_2$. Then the necessary condition for T^{**} is given by (33) (with $T_{\rm cr}$ replaced by T^{**}). We showed previously that this equation admits only one positive solution, that is, the existence of T^{**} implies that $T_{\rm cr}$ does not exist. Further, note that for $T = T_{\rm min}$ the corresponding optimal solution (t_1, t_2, t_3) of the problem (29)–(32) has the property $t_1 = t_2 = t_3$. Similarly, if for $T > T_{\rm min}$ this optimal solution satisfies $t_1 < t_2$, then $t_3 = t_2 + \frac{1}{k}\ln 2$. However, the mapping $\psi(T)$ is u.s.c.-B for $T \geq T_{\rm min}$, hence for $T > T_{\rm min}$, T being sufficiently close to $T_{\rm min}$, the optimal solution has to satisfy $t_1 = t_2$. Consequently, T^{**} cannot exist without the appearance of $T_{\rm cr}$ and this is a contradiction.

Summarizing the previous considerations we arrive at two cases described in the following theorem.

Theorem 15 *Let (t_1, t_2, t_3) be the optimal solution of the problem (29)–(32) and let Hypothesis 11 be fulfilled. Then either $t_1 = t_2$ for every $T \geq T_{min}$ or there exists a unique value of T_{cr} with the property that for $T \in [T_{min}, T_{cr}]$ the optimal solution satisfies $t_1 = t_2$ and for $T > T_{cr}$ the property $t_1 < t_2$ is fulfilled. Moreover, this value T_{cr} is found as the unique positive solution of Equation (33).*

Remark 16 The numerical calculations (Bazaraa et al. [2] give further information about useful algorithms) show that considering parameter T large enough the optimal solution (t_1, t_2, t_3) of the problem (29)–(32) satisfies $t_1 < t_2$ for given fixed parameters α, β, L and k. We therefore introduce a conjecture that the first variant described in Theorem 15 (that is, $t_1 = t_2$ for every $T \geq T_{\min}$) does not actually occur. However, the proof of this conjecture for arbitrary (unspecified) values of α, β, L and k remains open.

For the sake of simplicity, the results of this section so far have been illustrated by the model where resistance is a linear function of the speed of the train. The extension to models where resistance depends nonlinearly on the speed consists only in more tedious calculations and does not represent any qualitative advancement. Considering the quadratic resistance function we introduce and discuss the problem of the critical time T_{cr} similarly to the case of the linear resistance. However, the formal justification of the existence of T_{cr} would be much more complicated. Therefore, we shall show at least the derivation of the necessary condition for T_{cr}, that is, the analogy of Equation (33).

We use the above derived Equation (21), recall that this relation was derived under assumption $t_1 < t_2$. Letting $t_2 \to t_1^+$ we get $T \to T_{\mathrm{cr}}$. We therefore put $t_2 = t_1 = t_{\mathrm{cr}}$ in corresponding formulas and obtain

$$t_3 = t_{\mathrm{cr}} + \frac{1}{2\sqrt{\beta k} \tanh\left(\sqrt{\beta k} t_{\mathrm{cr}}\right)}\,.$$

We compare this expression of time t_3 with Equation (22) to obtain

$$\begin{aligned} T_{\mathrm{cr}} &= \frac{1}{\sqrt{\alpha k}} \arctan\left[\sqrt{\frac{\beta}{\alpha}\frac{2}{3}} \tanh\left(\sqrt{\beta k} t_{\mathrm{cr}}\right)\right] \\ &\quad + t_{\mathrm{cr}} + \frac{1}{2\sqrt{\beta k} \tanh\left(\sqrt{\beta k} t_{\mathrm{cr}}\right)}\,. \end{aligned} \tag{34}$$

This expression is substituted into Equation (21) and after some simple modifications we arrive at

$$\frac{2}{3}e^{kL}\left|\cos\arctan\left[\sqrt{\frac{\beta}{\alpha}}\frac{2}{3}\tanh\left(\sqrt{\beta k}t_{cr}\right)\right]\right| - \cosh\left(\sqrt{\beta k}t_{cr}\right) = 0\,.$$

This equation determines t_{cr} and from Equation (34) we obtain the value of T_{cr}.

6 Conclusion

This section summarizes the procedures leading to the complete solution of the problem (1)–(5) with respect to the parameter T. First, we calculate the value of T_{min} from the corresponding equation. If $T < T_{min}$, then the problem does not have any solution. If $T = T_{min}$ then it is necessary to use relations for time optimization. If $T > T_{min}$, we verify the validity of Hypothesis 11 and determine the value of T_{cr} from (33) or (34) (for the linear or quadratic resistance function, respectively). Now if $T_{min} < T \leq T_{cr}$, then the optimal strategy does not contain the speed holding phase $\hat{u}(t) = r(\hat{x}_2(t))$. Moreover, considering the linear resistance function we use Equation (11) and Equation (10) to calculate the values of $t_1 = t_2$ and t_3 (or Equation (13) and Equation (12) in the case of quadratic resistance). The singular control $\hat{u}(t) = r(\hat{x}_2(t))$ is involved in the optimal strategy solely in case $T > T_{cr}$ (more precisely, if this control level occurs for some $T > T_{cr}$, then it occurs for any $T > T_{cr}$). The values t_1, t_2 and t_3 are then determined via relations (19), (18) and (20) (the linear case) or via (21), (23) and (22) (the quadratic case).

To outline the previous procedures we present the following illustrative simple example. Consider the problem (1)–(5), where $r(x_2) = x_2$, $L = \alpha = \beta = 1$. First we calculate the value of T_{min} according to (6). We obtain $T_{min} = 2.170$. From (33) we arrive at the value $T_{cr} = 2.316$. Now, if we choose T such that $T_{min} < T < T_{cr}$, the switching times $t_1 = t_2$ and t_3 are easily calculated according to the Equation (11) and (10), respectively (for example, if $T = 2.2$, then $t_1 = t_2 = 1.445$ and $t_3 = 1.755$, the speed holding phase is omitted). If $T > T_{\mathrm{cr}}$, the optimal case involves the speed holding level; for example, if $T = 10$, we easily verify that the relations (19), (18) and (20) imply the optimal values $t_1 = 0.108$, $t_2 = 9.257$ and $t_3 = 9.950$.

Similarly, if $r(x_2) = (x_2)^2$ (the values of L, α, β remain unchanged), then $T_{\mathrm{min}} = 2.062$ and $T_{\mathrm{cr}} = 2.172$ (see (7) and (34), respectively). Now, if $T = 2.2 > T_{\mathrm{cr}}$, then (in contrast to the linear case) the speed holding phase is involved in the optimal solution and $t_1 = 0.963$, $t_2 = 1,068$, $t_3 = 1.739$.

The models discussed by Cheng et al. [3] or Howlett [6, 11] were more general than the model (1)–(5). However, the problem of the critical time presented in this article have not previously been published. More general models for this problem will be the matter of future research.

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