Moving boundary shallow water flow in a region with quadratic bathymetry

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Abstract

Exact solutions of the nonlinear shallow water wave equations for forced flow involving linear bottom friction in a region with quadratic bathymetry have been found. These solutions also involve moving shorelines. The motion decays over time. In the solution of the three simultaneous nonlinear partial differential shallow water wave equations it is assumed that the velocity is a function of time only and along one axis. This assumption reduces the three simultaneous nonlinear partial differential equations to two simultaneous linear ordinary differential equations. The analytical model has been tested against a numerical solution with good agreement between the numerical and analytical solutions. The analytical model is useful for testing the accuracy of a moving boundary shallow water numerical model.

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1 Introduction

Exact solutions of the nonlinear shallow water wave equations were found by Thacker [13] for unforced frictionless flow involving the Coriolis force in parabolic canals. The solutions involve moving shorelines. The motion is oscillatory and continues indefinitely over time. Sachdev, Paliannapan and Sarathy [9] built on Thacker's work.

Carrier and Greenspan [2] obtained moving boundary analytical solutions of the one dimensional nonlinear shallow water wave equations for motion caused by periodic forcing in a frictionless fluid above a bed of constant slope. Johns [4] expressed these results in a simpler form.

The work in this article builds on the work of Thacker [13] for unforced flow in a parabolic canal; Thacker's solutions were discussed in detail by Sampson, Easton and Singh [11]. There have been no other analytical solutions of the nonlinear shallow water wave equations as a consequence of the work of Thacker [13] apart from three previous articles by Sampson et al. [10, 11, 12] and the article by Sachdev, Paliannapan and Sarathy [9].

Balzano [1], Holdahl, Holden and Lie [3], Lewis and Adams [6], Peterson et al. [8] and Yoon and Cho [15] compared numerical solutions of the nonlinear shallow water wave equations with some of the analytical solutions by Thacker [13].

Exact solutions of the nonlinear shallow water wave equations for forced flow involving linear bottom friction and without the Coriolis force in a region with quadratic bathymetry are found. These solutions also involve moving shorelines. The motion decays over time. The exact solutions developed here are a modification of the solutions given by Sampson et al. [11].

The analytical model has been tested against a numerical solution with good agreement between the numerical and analytical solutions. The numerical model is adapted from the SLM (Selective Lumped Mass) numerical model of Kawahara, Hirano and Tsubota [5]. The wetting and drying scheme used is different to that in the SLM model. The SLM model is finite element in space, using fixed triangular elements, finite difference in time and is explicit.

2 Model equations

We consider the case where the motion of shallow water in a basin is governed by the equations [14]

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \tau U + g\frac{\partial \zeta}{\partial x} = 0\,, \qquad (1)$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \tau V + g\frac{\partial \zeta}{\partial y} = 0\,, \qquad (2)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h+\zeta)U}{\partial x} + \frac{\partial (h+\zeta)V}{\partial y} = 0\,, \qquad (3)$$

where $\zeta(x,y,t)$ is the height of the water surface above mean water level, $z=-h(x,y)$ is the bottom surface, $U(x,y,t)$ is the depth averaged velocity component of the water current to the East, $V(x, y, t)$ is the depth averaged velocity component of the water current to the North, g is the acceleration due to gravity, τ is the bottom friction parameter and t is the time. The bottom friction parameter, τ, is considered to be constant, which implies that the bottom friction force varies linearly with velocity. In tidal modelling the bottom friction parameter is usually taken to be proportional to the magnitude of the velocity but occasionally it is accurate to consider it a constant [7]. Equations (1) and (2) are equations of the form force equals mass times acceleration, where the force terms are friction and hydrostatic pressure, while (3) is a statement of mass conservation.

Equations (1), (2) and (3) differ from Thacker's in that whereas Thacker's equations included Coriolis force terms but did not include friction terms, equations (1), (2) and (3) do not include Coriolis force terms, but do include friction terms. Thacker [13] assumed that U and V were functions of t only.

Here we assume that

$$U = u_0(t)\,, \tag{4}$$
$$V = 0\,, \tag{5}$$

Then equations (1) and (2) together with equations (4) and (5) imply that

$$\zeta(x, y, t) = \zeta_0(t) + x\zeta_1(t)\,, \tag{6}$$

where (1) and (4) imply that

$$\zeta_1(t) = -\frac{1}{g}\left[\frac{du_0(t)}{dt} + \tau u_0(t)\right]. \tag{7}$$

Equation (16) determines $\zeta_0(t)$. Equation (6) implies that at any time t the water surface is a plane.

Section 3 considers flow in a region with quadratic bathymetry. The discussion is similar to that by Thacker [13], but because the shallow water equations (1) and (2) used in this article have a slightly different form to Thacker's and we made slightly different assumptions about the velocity's functional form the discussion leads to different conclusions.

3 Flow in a region with quadratic bathymetry

Assume that

$$h = h_0 \left(1 - \frac{x^2}{a^2}\right), \quad x \geq 0\,. \tag{8}$$

with h_0 and a constant, so that flow takes place in a region with quadratic bathymetry.

As was shown by Sampson et al. [11], substituting (4), (5), (6), and (8) in (3) and making use of (7) gives

$$\frac{d^2 u_0\,(t)}{dt^2} + \tau \frac{du_0\,(t)}{dt} + \frac{2gh_0 u_0\,(t)}{a^2} = 0\,, \tag{9}$$

and

$$\frac{d\zeta_0\,(t)}{dt} - \frac{1}{g} u_0(t) \frac{du_0}{dt} - \frac{\tau}{g} u_0(t)^2 = 0\,. \tag{10}$$

Equation (9) has to be solved for $u_0(t)$. As equation (9) is a second order differential equations, it requires two boundary conditions. The solution of (9) is substituted in (10), which is first order and hence needs one boundary condition to be solved uniquely for $\zeta_0(t)$.

The auxiliary equation for (9) is

$$\lambda^2 + \tau\lambda + \frac{2gh_0}{a^2} = 0\,. \tag{11}$$

The roots of (11) are

$$\lambda = \frac{-\tau \pm \sqrt{\tau^2 - p^2}}{2}\,. \tag{12}$$

where we define

$$p = \sqrt{\frac{8gh_0}{a^2}}\,. \tag{13}$$

Hence, the three possible solutions of (11) are for when $\tau < p$, $\tau > p$, and $\tau = p$. The solutions of (6), (7), (9) and (10) for $\tau < p$ only are discussed in equations (14)–(18).

If $\tau < p$, then the general solution of (9) is

$$u_0(t) = Ae^{-\tau t/2}\cos st + Be^{-\tau t/2}\sin st \tag{14}$$

where A and B are constants and where

$$s = \frac{\sqrt{p^2 - \tau^2}}{2}\,. \tag{15}$$

Equation (14) implies that $u_0(t) \to 0$ as $t \to \infty$.

Substituting (14) in (10) and integrating with respect to t gives

$$\begin{aligned} \zeta_0(t) &= \frac{a^2e^{-\tau t}}{8g^2h_0}\left\{\left[\frac{(B^2 - A^2)s + \tau AB}{2}\right](-\tau\sin 2st - 2s\cos 2st)\right\} \\ &\quad + \frac{a^2e^{-\tau t}}{8g^2h_0}\left[ABs + \frac{\tau(A^2 - B^2)}{4}\right](-\tau\cos 2st + 2s\sin 2st) \\ &\quad - \frac{(A^2 + B^2)e^{-\tau t}}{4g}\,, \end{aligned} \tag{16}$$

with the constant of integration being zero because of the assumption that $\zeta_0(t) \to 0$ as $t \to \infty$. Substituting (14) in (7) gives

$$\zeta_1(t) = -\frac{e^{-\tau t/2}}{g}\left[(-As\sin st + Bs\cos st) + \left(\frac{\tau}{2}\right)(A\cos st + B\sin st)\right]. \tag{17}$$

Substituting (16) and (17) into (6) gives

$$\begin{aligned} \zeta(x,t) &= \frac{a^2e^{-\tau t}}{8g^2h_0}\left\{\left[\frac{(B^2 - A^2)s + \tau AB}{2}\right](-\tau\sin 2st - 2s\cos 2st)\right\} \\ &\quad + \frac{a^2e^{-\tau t}}{8g^2h_0}\left[ABs + \frac{\tau(A^2 - B^2)}{4}\right](-\tau\cos 2st + 2s\sin 2st) \end{aligned}$$

$$- \frac{e^{-\tau t/2}}{g} \left[(-As \sin st + Bs \cos st) + \left(\frac{\tau}{2}\right) (A \cos st + B \sin st) \right] x$$
$$- \frac{(A^2 + B^2) e^{-\tau t}}{4g} . \tag{18}$$

Here we assume that the flow is subject to a forcing at $x = 0$:

$$\zeta(0,t) = \frac{a^2 B^2 e^{-\tau t}}{8 g^2 h_0} \left[-s\tau \sin 2st + \left(\frac{\tau^2}{4} - s^2 \right) \cos 2st \right] - \frac{B^2 e^{-\tau t}}{4g} . \tag{19}$$

where B is a constant.

The forcing at $x = 0$, as specified in (19), will be satisfied, as can be seen from (18), if $A = 0$ and $B \neq 0$, and then it follows from (14) and (18) that

$$u_0(t) = B e^{-\tau t/2} \sin st \,, \tag{20}$$

and

$$\begin{aligned} \zeta(x,t) \;\; = \;\; & \frac{a^2 B^2 e^{-\tau t}}{8 g^2 h_0} \left[-s\tau \sin 2st + \left(\frac{\tau^2}{4} - s^2 \right) \cos 2st \right] - \frac{B^2 e^{-\tau t}}{4g} \\ & - \frac{e^{-\tau t/2}}{g} \left(Bs \cos st + \frac{\tau B}{2} \sin st \right) x \,. \end{aligned} \tag{21}$$

At the shoreline, the total depth

$$h + \zeta = 0 \,. \tag{22}$$

Substituting (8) and (21) in (22) gives

$$x = \frac{a^2 e^{-\tau t/2}}{2 h_0 g} \left(-Bs \cos st - \frac{\tau B}{2} \sin st \right) + a \,. \tag{23}$$

Hence, the projection of the moving shoreline on the xy plane is a straight line.

The water moves backwards and forwards across the region of flow with motion dying out as $t \to \infty$. As $t \to \infty$ the shoreline approaches

$$x = a \tag{24}$$

the shoreline for an undisturbed surface, and $\zeta \to 0$, so that the motion will eventually die out.

4 The analytical solution versus the numerical solution

The analytical model has been tested against a numerical solution with good agreement between the numerical and analytical solutions. As the shoreline moves over time, in the numerical solution there will be some nodes that are wet part of the time and dry part of the time. The numerical model is adapted from the SLM (Selective Lumped Mass) numerical model of Kawahara et al. [5]. The wetting and drying scheme used, discussed in detail by Sampson et al. [12], is different to that in the SLM model. The SLM model is finite element in space, using fixed triangular elements, finite difference in time and is explicit.

For the numerical model the values chosen were $h_0 = 10\,\mathrm{m}$, $a = 3000\,\mathrm{m}$, $\tau = 0.001\,\mathrm{s}^{-1}$ and $B = 2\,\mathrm{ms}^{-1}$ with the initial values of ζ and U set to those of the analytical model. The period of the trigonometric terms in the motion, T, is 1353 s. The initial velocity is $0\,\mathrm{ms}^{-1}$. At the open water boundary, at $x = 0$, the water level was specified as the same function of time as in the analytical model. The calculation, using a program written in Visual C++, was done over eight periods (10827 s).

A triangular mesh was used, covering a rectangular region of width 4320 m in the x direction and height 240 m in the y direction. Each triangle in the mesh is an isosceles right angled triangle. The mesh contains 4913 nodes and 9216 elements. In the calculations, the time step δt had to be set to less than or equal to 0.195 s for convergence. The values of water elevation, ζ, and the shoreline discussed below are for nodes sitting on a line parallel to the base of the rectangular region and half way between the base and top of the region.

A plot of the numerical and analytical x-coordinates of the shoreline as a function of time over one period, T, is shown in Figure 1. The analytical solution is shown in each diagram as a continuous curve while the numerical solution is a number of points; these points are so close together that they appear to be a number of straight lines parallel to the time axis. As the distance between successive nodes is 15 m, the distance between successive apparent straight lines is 15 m, which means that numerically when the shoreline moves it moves 15 m in one time step. There is good agreement between the analytical and numerical values.

A graphical comparison of the numerical and analytical values for the water level, ζ, against x at time $t = T/2$ is shown respectively in Figures 2. The values are close.

5 Conclusions

Exact solutions of the one dimensional nonlinear shallow water wave equations in the case of forced flow involving bottom friction and without the Coriolis force in a region with quadratic bathymetry have been found. These solutions also involve moving shorelines. The motion decays over time as expected in a motion involving friction and an input force that decays over time. In contrast, Thacker found exact solutions of the two dimensional nonlinear shallow water wave equations in the case of unforced frictionless flow involving the Coriolis force in a parabolic canal. These solutions also involve moving shorelines. The motion is oscillatory and continues indefinitely over time as expected for motion involving no friction.

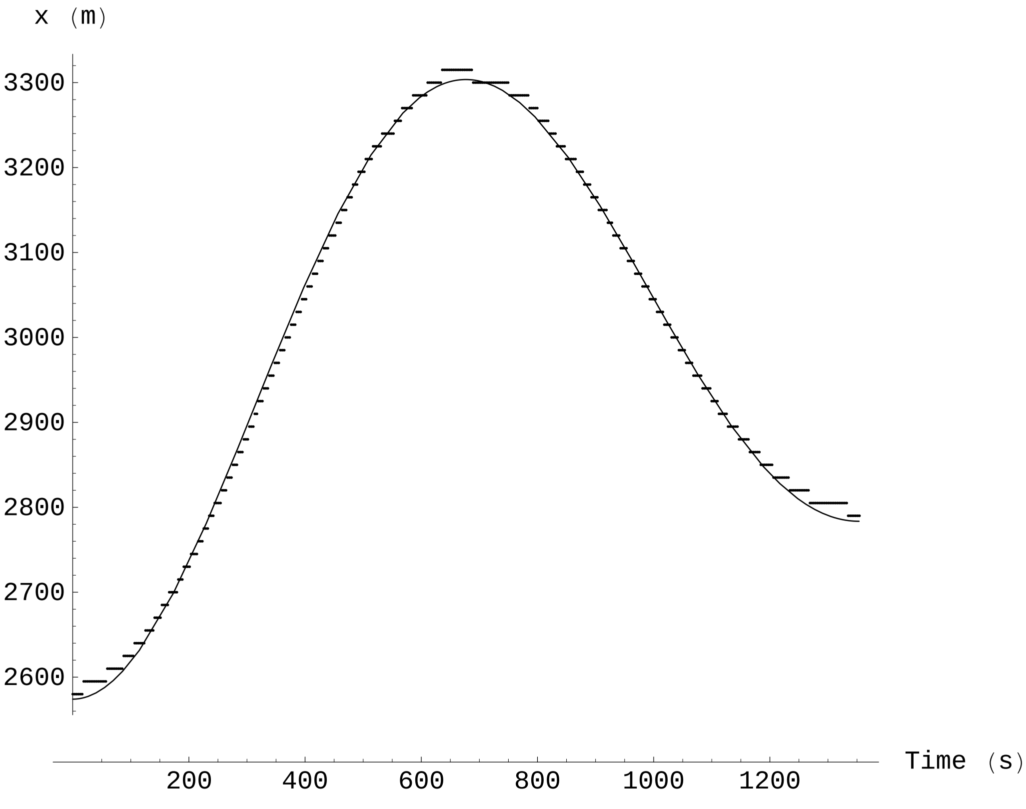

Figure 1: A plot of the numerical and analytical values of the x-coordinate of the shoreline as a function of time over one period. The analytical solution is a continuous curve while the numerical solution is a number of points.

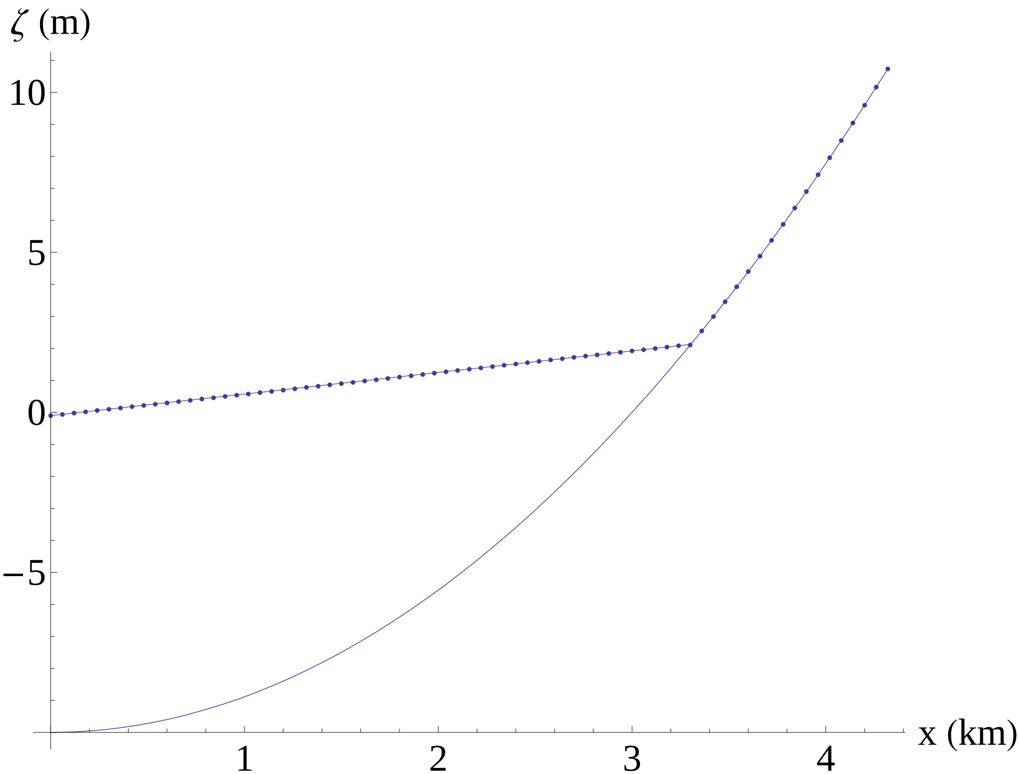

Figure 2: A comparison of the numerical and analytical values of the water surface at time $t = T/2$. The analytical solution is a continuous line whereas the numerical solution is a series of dots; the results for every fourth node are shown.

The solutions found are useful for testing numerical solutions of the nonlinear shallow water wave equations which include bottom friction and whose flow involves moving shorelines. The analytical model has been tested against a numerical solution with good agreement between the numerical and analytical solutions.

References

[1] Balzano, A., Evaluation of methods for numerical simulation of wetting and drying in shallow water flow models, *Coastal Engineering*, **34**, 1998, 83–107. C668

[2] Carrier, G. F. and Greenspan, H. P., Water waves of finite amplitudes on a sloping beach, *Journal of Fluid Mechanics*, **4**, 1958, 97–109. C667

[3] Holdahl, R., Holden, H., and Lie,K-A., Unconditionally Stable Splitting Methods For the Shallow Water Equations, *BIT*, **39**, 1998, 451–472. C668

[4] Johns, B., Numerical integration of the shallow water equations over a sloping shelf, *International Journal for Numerical Methods in Fluids*, **2**, 1982, 253–261. C667

[5] Kawahara, M., Hirano, H., and Tsubota, K., Selective lumping finite element method for shallow water flow, *International Journal for Numerical Methods in Fluids*, **2**, 1982, 89–112. C668, C673

[6] Lewis, C. H. III and Adams, W. M., Development of a tsunami-flooding model having versatile formation of moving boundary conditions, *The Tsunami Society Monograph Series*, 1983, No. 1, 128 pp. C668

[7] Parker, B. B., *Frictional Effects on the Tidal Dynamics of a Shallow Estuary*, PhD thesis, The John Hopkins University, 1984. C669

[8] Peterson P., Hauser J., Thacker W. C., Eppel D., An Error-Minimizing Algorithm for the Non-Linear Shallow-Water Wave Equations with Moving Boundaries. In *Numerical Methods for Non-Linear Problems*, editors C. Taylor, E. Hinton, D. R. J. Owen and E. Onate, 2, Pineridge Press, 1984, 826–836, http://www.cle.de/hpcc/publications/ C668

[9] Sachdev, P. L., Paliannapan, D. and Sarathy, R., Regular and chaotic flows in paraboloidal basins and eddies, *Chaos, Solitons and Fractals*, **7**, 1996, 383–408 . C667

[10] Sampson, J., Easton, A., and Singh, M., Moving Boundary Shallow Water Flow in Circular Paraboloidal Basins. *Proceedings of the Sixth Engineering Mathematics and Applications Conference, 5th International Congress on Industrial and Applied Mathematics, at the University of Technology, Sydney, Australia*, editors R. L. May and W. F. Blyth, 2003, 223–227. C667

[11] Sampson, J., Easton, A., and Singh, M., Moving boundary shallow water flow in parabolic bottom topography, *Australian and New Zealand Industrial and Applied Mathematics Journal*, **47 (EMAC2005)**, C373–C387, 2006, http://anziamj.austms.org.au/V47EMAC2005/Sampson C667, C668, C670

[12] Sampson, Joe, Easton, Alan and Singh, Manmohan, A New Moving Boundary Shallow Water Wave Numerical Model, *Australian and New Zealand Industrial and Applied Mathematics Journal*, **48 (CTAC2006)**, C605–C617, 2007, http://anziamj.austms.org.au/ojs/index.php/ANZIAMJ/article/view/78 C667, C673

[13] Thacker, W. C., Some exact solutions to the nonlinear shallow-water wave equations, *J. Fluid. Mech.*, **107**, 1981, 499–508. C667, C668, C669

[14] Vreugdenhil, C. B., *Numerical Methods for Shallow-Water Flow*, Kluwer Academic Publishers, 1998. C668

[15] Yoon S. B., and Cho J. H., Numerical simulation of Coastal Inundation over Discontinuous Topography, *Water Engineering Research*, **2**(2), 2001, 75–87 C668

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