Monotone iterates for solving systems of semilinear elliptic equations and applications

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Abstract

Consider monotone finite difference iterative algorithms for solving coupled systems of semilinear elliptic equations. A monotone domain decomposition algorithm based on a modification of Schwarz alternating method and on decomposition of a computational domain into nonoverlapping subdomains is constructed. Advantages of the algorithm are that the algorithm solves only linear discrete systems at each iterative step, converges monotonically to the exact solution of the nonlinear discrete problem, and is potentially parallelisable. The monotone domain decomposition algorithm is applied to a gas-liquid interaction model. Numerical experiments confirm theoretical results.

Contents

1   Introduction

The elliptic system under consideration is in the form

$$-\nabla \cdot (a_i \nabla u_i) + f_i(x, u_1, u_2) = 0, \quad x \in \omega, \quad i = 1, 2, \tag{1}$$

$$u_i(x) = g_i(x), \quad i = 1, 2, \quad x \in \partial\omega,$$

$$x \in \mathbb{R}^k, \quad \bar{\omega} = \prod_{\alpha=1}^{k} \bar{\omega}^{x_\alpha}, \quad \bar{\omega}^{x_\alpha} = \{0 \leq x_\alpha \leq r_\alpha\},$$

where $\bar{\omega} = \omega \cup \partial\omega$, $\partial\omega$ is the boundary and ∇ is gradient operator in ω. We assume that $a_i(x) > 0$, $i = 1, 2$, on $\bar{\omega}$, and a_i, f_i and g_i, $i = 1, 2$, are sufficiently smooth functions.

In solving such nonlinear problems by the finite difference method, the corresponding discrete problem is usually formulated as a system of nonlinear algebraic equations. One then requires a reliable and efficient computational algorithm for computing the solution. A fruitful method for the treatment of these nonlinear systems is the method of upper and lower solutions and its associated monotone iterations [3].

Iterative domain decomposition algorithms based on Schwarz-type alternating procedures have received much attention for their potential as efficient

algorithms for parallel computing. For solving nonlinear elliptic equations, Boglaev [1, 2] proposed the discrete iterative algorithms which combine the monotone approach and an iterative domain decomposition method based on the Schwarz alternating procedure. The computational domain is partitioned into many nonoverlapping subdomains with interface γ. Small interfacial subdomains are introduced near the interface γ, and approximate boundary values computed on γ are used for solving problems on nonoverlapping subdomains. Thus, this approach may be considered as a variant of a block Gauss–Seidel iteration (or in the parallel context as a multi-coloured algorithm) for the subdomains with a Dirichlet–Dirichlet coupling through the interface variables. This article extends the monotone domain decomposition algorithms of Boglaev [1, 2] to the system of nonlinear elliptic equations (1).

Section 2 introduces a nonlinear finite difference scheme for the numerical solution of (1). Section 3 presents a monotone domain decomposition algorithm. We show that monotonic convergence is maintained under the proposed domain decomposition into nonoverlapping subdomains and associated algorithm. Section 4 applies the monotone domain decomposition to a gas-liquid interaction model.

2   A nonlinear difference scheme

On $\bar{\omega}^{x_\alpha}$, $\alpha = 1, \ldots, k$, we set up nonuniform rectangular meshes

$$\bar{\omega}^{hx_\alpha} = \{x_\alpha^{(i_\alpha)}, \quad 0 \leq i_\alpha \leq N_\alpha; \quad x_\alpha^{(0)} = 0, \quad x_\alpha^{(N_\alpha)} = r_\alpha\}, \quad \alpha = 1, \ldots, k.$$

Thus, we represent the mesh $\bar{\omega}^h$ in the form

$$\bar{\omega}^h = \prod_{\alpha=1}^{k} \bar{\omega}^{hx_\alpha}, \quad \bar{\omega}^h = \omega^h \cup \partial\omega^h,$$

where ω^h and $\partial\omega^h$ are sets of interior and boundary mesh points, respectively.

For solving (1), consider the nonlinear difference scheme in the canonical form [5]

$$\mathcal{L}_i^h U_i(p) + f_i(p, U_1, U_2) = 0, \quad p \in \omega^h, \quad U_i = g_i \text{ on } \partial\omega^h, \quad i = 1, 2, \quad (2)$$
$$\mathcal{L}_i^h U_i(p) \equiv d_i(p) U_i(p) - \sum_{p' \in \sigma'(p)} e_i(p, p') U_i(p'),$$

where $\sigma'(p) = \sigma(p) \setminus \{p\}$, $\sigma(p)$ is a stencil of the scheme at an interior mesh point $p \in \omega^h$ and $\partial\omega^h$ is the boundary of $\bar{\omega}^h$. We make the following assumptions on the coefficients of the difference operators $\mathcal{L}_i^h$, $i = 1, 2$:

$$d_i(p) > 0, \quad e_i(p, p') \geq 0, \quad d_i(p) - \sum_{p' \in \sigma'(p)} e_i(p, p') \geq 0, \quad p \in \omega^h. \quad (3)$$

We now formulate the discrete maximum principle.

Lemma 1 *Let the coefficients of the difference operators $\mathcal{L}_i^h$, $i = 1, 2$, in (2) satisfy (3). If mesh functions $W_i(p)$, $i = 1, 2$, satisfy the conditions*

$$(\mathcal{L}_i^h + c_i) W_i(p) \geq 0 \ (\leq 0), \quad p \in \omega^h, \quad W_i(p) \geq 0 \ (\leq 0), \quad p \in \partial\omega^h,$$

where $c_i(p) \geq 0$, $i = 1, 2$, then $W_i(p) \geq 0 \ (\leq 0)$, $i = 1, 2$, in $\bar{\omega}^h$.

Samarskii [5] gave the proof of this lemma.

Remark 2 Difference schemes which satisfy the maximum principle from Lemma 1 are said to be monotone. The monotonicity condition guarantees that the systems of algebraic equations based on such methods are well-posed.

3   Monotone domain decomposition algorithm

By hyperplane

$$\{x_1 = \rho_m, m = 1, \ldots, M - 1 : \rho_0 = 0 < \rho_1 < \cdots < \rho_{M-1} < \rho_M = r_1\},$$

we decompose the mesh $\bar{\omega}^h$ into M nonoverlapping rectangular subdomains $\bar{\omega}^h_m$, $m = 1, \ldots, M$:

$$\bar{\omega}^h = \bigcup_{m=1}^{M} \bar{\omega}^h_m, \quad \partial\omega^h_m = \gamma^0_m \cup \gamma_{m-1} \cup \gamma_m, \quad \gamma^0_m = \partial\omega^h \cap \bar{\omega}^h_m,$$

$$\gamma_m = \{\rho_m\} \times \bar{\omega}^{hy}, \quad \bar{\omega}^{hy} \equiv \prod_{\alpha=2}^{k} \bar{\omega}^{hx_\alpha}, \quad \bar{\omega}^h_m \cap \bar{\omega}^h_{m+1} = \gamma_m,$$

where the boundary $\partial\omega^h_m$ of $\bar{\omega}^h_m$ consists of the boundaries γ_{m-1} and γ_m which belong to the hyperplane ρ_{m-1} and ρ_m, respectively, and γ^0_m which belongs to the boundary $\partial\omega^h$. On $\bar{\omega}^h$ we introduce $(M - 1)$ interfacial subdomains $\bar{\vartheta}^h_m$, $m = 1, \ldots, M - 1$, with boundaries $\partial\vartheta^h_m$ in the form

$$\partial\vartheta^h_m = \gamma^c_m \cup \gamma^b_m \cup \gamma^e_m, \quad \gamma^c_m = \partial\omega^h \cap \bar{\vartheta}^h_m,$$

$$\gamma^b_m = \{\rho^b_m\} \times \bar{\omega}^{hy}, \quad \gamma^e_m = \{\rho^e_m\} \times \bar{\omega}^{hy}, \quad \rho^b_m < \rho_m < \rho^e_m,$$

where $\bar{\vartheta}^h_m$ overlaps $\bar{\omega}^h_m \cup \bar{\omega}^h_{m+1}$. Figure 1 illustrates the domain decomposition in the two dimensional case.

We assume that sizes of all the subdomains $\bar{\omega}^h_m$ and $\bar{\vartheta}^h_m$ allow us to solve Dirichlet boundary value problems based on the difference equations from (2).

3.1   Statement of domain decomposition algorithm

Consider the following domain decomposition approach for solving (2). On each iterative step, we first solve problems on the nonoverlapping subdo-
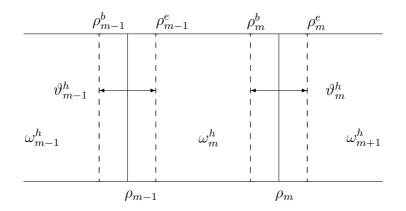
FIGURE 1: Fragment of the domain decomposition.

mains $\bar{\omega}_m^h$, $m = 1, \ldots, M$, with Dirichlet boundary conditions passed from the previous iterate. Then Dirichlet data are passed from these subdomains to the interfacial subdomains $\bar{\vartheta}_m^h$, $m = 1, \ldots, M-1$, and problems on the interfacial subdomains are computed. Finally, we piece together the solutions on the subdomains.

1. Initialization: On the whole mesh $\bar{\omega}^h$, choose initial functions $V_i^{(0)}(p)$, $p \in \bar{\omega}^h$, $i = 1, 2$.

2. On subdomains $\bar{\omega}_m^h, m = 1, \ldots, M$, compute mesh functions $V_{i,m}^{(n)}(p)$, $m = 1, \ldots, M$, $i = 1, 2$, (here the index n stands for a number of iterative steps) satisfying the difference schemes

$$(\mathcal{L}_i^h + \bar{c}_i)Z_{i,m}^{(n)}(p) = -\mathcal{R}_i^h(p, V_1^{(n-1)}, V_2^{(n-1)}), \quad p \in \omega_m^h, \qquad (4)$$

$$Z_{i,m}^{(1)}(p) = \begin{cases} g_i(p) - V_i^{(0)}(p), & p \in \gamma_m^0, \\ 0, & p \in \gamma_{m-1} \cup \gamma_m, \end{cases}$$

$$Z_{i,m}^{(n)}(p) = 0, \quad p \in \partial\omega_m^h, \quad n \geq 2,$$

$$\mathcal{R}_i^h(p, V_1^{(n-1)}, V_2^{(n-1)}) \equiv \mathcal{L}_i^h V_i^{(n-1)} + f_i(p, V_1^{(n-1)}, V_2^{(n-1)}),$$

where mesh functions $\bar{c}_i(p)$, $i = 1, 2$, is defined in (9).

3. On the interfacial subdomains $\bar{\vartheta}_m^h$, $m = 1, \ldots, M - 1$, compute the difference problems

$$(\mathcal{L}_i^h + \bar{c}_i)Y_{i,m}^{(n)}(p) = -\mathcal{R}_i^h(p, V_1^{(n-1)}, V_2^{(n-1)}), \quad p \in \vartheta_m^h, \qquad (5)$$

with the boundary conditions $Y_{i,m}^{(n)}(\gamma_m^c) = 0$, $Y_{i,m}^{(n)}(\gamma_m^b) = Z_{i,m}^{(n)}(\gamma_m^b)$, $Y_{i,m}^{(n)}(\gamma_m^e) = Z_{i,m+1}^{(n)}(\gamma_m^e)$.

4. Compute the mesh functions $V_i^{(n)}(p)$, $p \in \bar{\omega}^h$, $i = 1, 2$, by piecing together the solutions on the subdomains

$$V_i^{(n)}(p) = \begin{cases} V_{i,m}^{(n)}(p), & p \in \bar{\omega}_m^h \setminus (\bar{\vartheta}_{m-1}^h \cup \bar{\vartheta}_m^h), \quad m = 1, \ldots, M, \\ W_{i,m}^{(n)}(p), & p \in \bar{\vartheta}_m^h, \quad m = 1, \ldots, M - 1, \end{cases} \qquad (6)$$

where $V_{i,m}^{(n)} = V_i^{(n-1)} + Z_{i,m}^{(n)}$, $W_{i,m}^{(n)} = V_i^{(n-1)} + Y_{i,m}^{(n)}$, and $\bar{\vartheta}_0^h = \emptyset$, $\bar{\vartheta}_M^h = \emptyset$.

5. Stopping criterion is

$$\max_{p \in \bar{\omega}^h} |V_i^{(n)}(p) - V_i^{(n-1)}(p)| \leq \delta, \quad i = 1, 2,$$

where δ is a prescribed accuracy. If the stopping criterion is reached, then stop, otherwise go to Step 2.

Algorithm (4)–(6) can be carried out by parallel processing, since on each iterative step n the $2M$ problems (4) for $Z_{i,m}^{(n)}(p)$, $m = 1, \ldots, M$, $i = 1, 2$, and the $2(M - 1)$ problems (5) for $Y_{i,m}^{(n)}(p)$, $m = 1, \ldots, M - 1$, $i = 1, 2$, can be implemented concurrently.

3.2   Monotone convergence of algorithm (4)–(6)

We assume that the functions f_i, $i = 1, 2$, are quasi-monotone nonincreasing

$$- \partial f_1/\partial u_2 \le 0, \quad -\partial f_2/\partial u_1 \le 0. \tag{7}$$

Remark 3 Systems of elliptic equations with quasi-monotone nonincreasing reaction functions appear in a number of reaction-diffusion models including gas-liquid interaction models, the Belousov–Zhabotinskii reaction-diffusion model and the Volterra–Lotka competition model in ecology [4].

Vector mesh functions $\widetilde{\mathbf{U}}(p) = (\widetilde{U}_1(p), \widetilde{U}_2(p))$ and $\widehat{\mathbf{U}}(p) = (\widehat{U}_1(p), \widehat{U}_2(p))$ are called ordered upper and lower solutions of (2) if they satisfy the relation $\widetilde{\mathbf{U}}(p) \ge \widehat{\mathbf{U}}(p)$, $p \in \bar{\omega}^h$, and if

$$\mathcal{L}_1^h \widehat{U}_1(p) + f_1(p, \widehat{U}_1, \widetilde{U}_2) \le 0 \le \mathcal{L}_1^h \widetilde{U}_1(p) + f_1(p, \widetilde{U}_1, \widehat{U}_2), \quad p \in \omega^h, \tag{8}$$
$$\mathcal{L}_2^h \widehat{U}_2(p) + f_2(p, \widetilde{U}_1, \widehat{U}_2) \le 0 \le \mathcal{L}_2^h \widetilde{U}_2(p) + f_2(p, \widehat{U}_1, \widetilde{U}_2), \quad p \in \omega^h,$$
$$\widehat{U}_i \le g_i \le \widetilde{U}_i \quad \text{on } \partial\omega^h, \quad i = 1, 2.$$

The ordering relation $\widetilde{\mathbf{U}} \ge \widehat{\mathbf{U}}$ is meant in the componentwise sense, that is, $\widetilde{U}_i(p) \ge \widehat{U}_i(p)$, $i = 1, 2$.

We assume that f_i, $i = 1, 2$, satisfy the constraints

$$\max\{\partial f_i(p, U_1, U_2)/\partial u_i; \ \widehat{\mathbf{U}}(p) \le \mathbf{U}(p) \le \widetilde{\mathbf{U}}(p)\} \le \bar{c}_i(p), \tag{9}$$

where $\mathbf{U} = (U_1, U_2)$ and $\bar{c}_i(p) \ge 0$, $i = 1, 2$, are bounded functions in $\bar{\omega}^h$.

We get the following convergence property of algorithm (4)–(6).

Theorem 4 *Assume that the coefficients of the difference operators $\mathcal{L}_i^h$, $i = 1, 2$, in (2) satisfy (3), the functions f_i, $i = 1, 2$, satisfy (7) and (9). Let $\widetilde{\mathbf{V}}$ and $\widehat{\mathbf{V}}$ be ordered upper and lower solutions (8) of (2).*

If $(\bar{V}_1^{(0)}, \underline{V}_2^{(0)}) = (\widetilde{V}_1, \widehat{V}_2)$, then the sequence $\{\bar{V}_1^{(n)}, \underline{V}_2^{(n)}\}$ generated by (4)–(6) converges monotonically to a solution $(\bar{V}_1, \underline{V}_2)$ of (2):

$$\bar{V}_1 \leq \bar{V}_1^{(n+1)} \leq \bar{V}_1^{(n)}, \quad \underline{V}_2^{(n)} \leq \underline{V}_2^{(n+1)} \leq \underline{V}_2, \quad in \ \bar{\omega}^h.$$

If $(\underline{V}_1^{(0)}, \bar{V}_2^{(0)}) = (\widehat{V}_1, \widetilde{V}_2)$, then the sequence $\{\underline{V}_1^{(n)}, \bar{V}_2^{(n)}\}$ generated by (4)–(6) converges monotonically to a solution $(\underline{V}_1, \bar{V}_2)$ of (2):

$$\underline{V}_1^{(n)} \leq \underline{V}_1^{(n+1)} \leq \underline{V}_1, \quad \bar{V}_2 \leq \bar{V}_2^{(n+1)} \leq \bar{V}_2^{(n)}, \quad in \ \bar{\omega}^h.$$

Moreover, the corresponding iterations $(\bar{V}_1^{(n)}, \bar{V}_2^{(n)})$ and $(\underline{V}_1^{(n)}, \underline{V}_2^{(n)})$ are ordered upper and lower solutions.

Proof: Suppose that $(\bar{V}_1^{(0)}, \underline{V}_2^{(0)}) = (\widetilde{V}_1, \widehat{V}_2)$. By the maximum principle in Lemma 1, from (4) and (8), we conclude that

$$Z_{1,m}^{(1)}(p) \leq 0, \quad Z_{2,m}^{(1)}(p) \geq 0, \quad p \in \bar{\omega}_m^h. \tag{10}$$

Using the mean value theorem, from (4) we obtain for $i = 1, 2$,

$$\mathcal{R}_i^h(p, V_{1,m}^{(1)}, V_{2,m}^{(1)}) = -(\bar{c}_i - \partial f_i^{(1)}/\partial u_i) Z_{i,m}^{(1)} + (\partial f_i^{(1)}/\partial u_{i'}) Z_{i',m}^{(1)}, \quad p \in \omega_m^h, \tag{11}$$

where $i' \neq i$, the variable p is suppressed and the partial derivatives are calculated at intermediate points $(\eta_{i,m}, \mu_{i,m})$ such that $V_{1,m}^{(1)} \leq \eta_{i,m} \leq \bar{V}_1^{(0)}$, $\underline{V}_2^{(0)} \leq \mu_i \leq V_{2,m}^{(1)}$. From here, (7), (9) and (10), we conclude

$$\mathcal{R}_1^h(p, V_{1,m}^{(1)}, V_{2,m}^{(1)}) \geq 0, \quad \mathcal{R}_2^h(p, V_{1,m}^{(1)}, V_{2,m}^{(1)}) \leq 0, \quad p \in \omega_m^h, \tag{12}$$

$$V_{i,m}^{(1)}(p) = g_i(p), \quad p \in \gamma_m^0, \qquad V_{i,m}^{(1)}(p) = V_i^{(0)}(p), \quad p \in \gamma_{m-1} \cup \gamma_m,$$

where $V_1^{(0)} = \bar{V}_1^{(0)}$ and $V_2^{(0)} = \underline{V}_2^{(0)}$. Taking into account (10), by the maximum principle in Lemma 1, from (5) and (8), it follows that

$$Y_{1,m}^{(1)}(p) \le 0, \quad Y_{2,m}^{(1)}(p) \ge 0, \quad p \in \bar{\vartheta}_m^h. \tag{13}$$

Similar to (12), we can prove

$$\mathcal{R}_1^h(p, W_{1,m}^{(1)}, W_{2,m}^{(1)}) \ge 0, \quad \mathcal{R}_2^h(p, W_{1,m}^{(1)}, W_{2,m}^{(1)}) \le 0, \quad p \in \vartheta_m^h, \tag{14}$$

where $W_{i,m}^{(1)}(\gamma_m^c) = g_i(\gamma_m^c)$, $W_{i,m}^{(1)}(\gamma_m^b) = V_{i,m}^{(1)}(\gamma_m^b)$, $W_{i,m}^{(1)}(\gamma_m^e) = V_{i,m+1}^{(1)}(\gamma_m^e)$. From here, (12) and the definition of $V_i^{(1)}$, $i = 1, 2$, we conclude that

$$\mathcal{R}_1^h(p, V_1^{(1)}, V_2^{(1)}) \ge 0, \quad \mathcal{R}_2^h(p, V_1^{(1)}, V_2^{(1)}) \le 0, \quad p \in \omega^h \setminus \gamma,$$

where $\gamma = \bigcup_{m=1}^{M-1} \gamma_m^{b,e}$. From the boundary conditions for $V_{i,m}^{(1)}$ and $W_{i,m}^{(1)}$, $i = 1, 2$, it follows that $V_i^{(1)}$, $i = 1, 2$, satisfy the boundary conditions in (2). We now prove that the above inequalities hold true on the interfacial boundaries γ_m^b and γ_m^e, $m = 1, \ldots, M-1$. We check these inequalities in the case of the left boundary γ_m^b, since the second case is checked in a similar way. From (4), (5), (10) and (13), we conclude that the mesh functions $R_{i,m}^{(1)} = V_{i,m}^{(1)} - W_{i,m}^{(1)}$, $i = 1, 2$, satisfy the difference problems

$$(\mathcal{L}_i^h + \bar{c}_i) R_{i,m}^{(1)}(p) = 0, \quad p \in \vartheta_m^{hb} = \omega_m^h \cap \vartheta_m^h,$$
$$R_{i,m}^{(1)}(p) = 0, \quad p \in \partial \vartheta_m^{hb} \setminus \gamma_m, \quad R_{1,m}^{(1)}(p) \ge 0, \quad R_{2,m}^{(n)}(p) \le 0, \quad p \in \gamma_m.$$

In view of the maximum principle in Lemma 1,

$$V_{1,m}^{(1)}(p) - W_{1,m}^{(1)}(p) \ge 0, \quad V_{2,m}^{(1)}(p) - W_{2,m}^{(1)}(p) \le 0, \quad p \in \bar{\vartheta}_m^{hb}, \quad i = 1, 2. \tag{15}$$

From here, (3), (6), (12), and taking into account that $W_{i,m}^{(n)}(p) = V_{i,m}^{(n)}(p)$, $p \in \gamma_m^b$, $i = 1, 2$, it follows that

$$\mathcal{R}_1^h(p, V_1^{(1)}, V_2^{(1)}) \ge \mathcal{R}_1^h(p, V_{1,m}^{(1)}, V_{2,m}^{(1)}) \ge 0, \quad p \in \gamma_m^b,$$

$$\mathcal{R}_2^h(p, V_1^{(1)}, V_2^{(1)}) \le \mathcal{R}_2^h(p, V_{1,m}^{(1)}, V_{2,m}^{(1)}) \le 0 \,, \quad p \in \gamma_m^b \,.$$

This leads to the fact that $\bar{V}_1^{(1)}$ and $\underline{V}_2^{(1)}$ are upper and lower solutions of problem (2), respectively. From (10) and (13), we have $\bar{V}_1^{(1)} \le \bar{V}_1^{(0)}$, $\underline{V}_2^{(1)} \ge \underline{V}_2^{(0)}$. By induction on n, we can prove that $\{\bar{V}_1^{(n)}\}$ and $\{\underline{V}_2^{(n)}\}$ are monotonically decreasing and increasing sequences of upper and lower solutions to problem (2), respectively.

Similarly, for $(\underline{V}_1^{(0)}, \bar{V}_2^{(0)}) = (\widehat{V}_1, \widetilde{V}_2)$, we can prove that $\{\underline{V}_1^{(n)}\}$ and $\{\bar{V}_2^{(n)}\}$ are monotonically increasing and decreasing sequences of lower and upper solutions to problem (2), respectively.

We now prove that $(\bar{V}_i^{(n)}, \underline{V}_i^{(n)})$, $i = 1, 2$, are ordered upper and lower solutions

$$\underline{V}_i^{(n)}(p) \le \bar{V}_i^{(n)}(p) \,, \quad p \in \bar{\omega}^h \,, \quad i = 1, 2 \,. \tag{16}$$

Introduce the notation $Q_i^{(n)} = \bar{V}_i^{(n)} - \underline{V}_i^{(n)}$, $i = 1, 2$. Similar to (11), using the mean-value theorem, from (4)–(6) with $i = 1$ and $n = 1$, (7), (9) and taking into account that $\widetilde{\mathbf{V}}$ and $\widehat{\mathbf{V}}$ are ordered upper and lower solutions to (2), we obtain

$$(\mathcal{L}_1^h + \bar{c}_1)Q_1^{(1)} = (\bar{c}_i - \partial f_1^{(1)}/\partial u_1)Q_1^{(0)} + (\partial f_1^{(1)}/\partial u_2)Q_2^{(0)} \ge 0 \,, \quad p \in \omega^h \setminus \gamma \,, \tag{17}$$

$$Q_1^{(1)}(p) = 0 \,, \quad p \in \partial\omega^h.$$

We prove that

$$(\mathcal{L}_1^h + \bar{c}_1)Q_1^{(1)} \ge 0 \,, \quad p \in \gamma \,.$$

We check this inequality on γ_m^b. Similar to (17), in the notation $Q_{i,m}^{(n)} = \bar{V}_{i,m}^{(n)} - \underline{V}_{i,m}^{(n)}$, we can get

$$(\mathcal{L}_1^h + \bar{c}_1)Q_{1,m}^{(1)} = (\bar{c}_i - \partial f_1^{(1)}/\partial u_1)Q_1^{(0)} + (\partial f_1^{(1)}/\partial u_2)Q_2^{(0)} \ge 0 \,, \quad p \in \omega_m^h.$$

From here, (3), (6), (12), (15), the inverse inequalities in (15) for the case $(\underline{V}_1^{(0)}, \bar{V}_2^{(0)}) = (\widehat{V}_1, \widetilde{V}_2)$, and taking into account that $W_{i,m}^{(n)}(p) = V_{i,m}^{(n)}(p)$, $p \in$

γ_m^b, $i = 1, 2$, it follows that

$$(\mathcal{L}_1^h + \bar{c}_1)Q_1^{(1)} \geq (\mathcal{L}_1^h + \bar{c}_1)Q_{1,m}^{(1)} \geq 0\,, \quad p \in \gamma_m^b\,.$$

From here and (17), by the maximum principle in Lemma 1, we conclude (16) for $i = 1$, $n = 1$. Now, by induction on n, we can prove (16) for $i = 1$. In the same way, we can prove (16) for $i = 2$ and $n \geq 1$.

From (16), we conclude that the monotonically decreasing and increasing sequences $\{\bar{V}_i^{(n)}\}$, $\{\underline{V}_i^{(n)}\}$, $i = 1, 2$, are bounded from below and above, respectively, by any lower and upper solutions, and, hence, are convergent

$$\lim_{n \to \infty} \bar{V}_i^{(n)}(p) = \bar{V}_i(p)\,, \quad \lim_{n \to \infty} \underline{V}_i^{(n)}(p) = \underline{V}_i(p)\,, \quad p \in \bar{\omega}^h.$$

From here, linearity of $\mathcal{L}_i^h$, $i = 1, 2$, and continuity of f_i, $i = 1, 2$, we conclude that $(\bar{V}_1, \underline{V}_2)$ and $(\underline{V}_1, \bar{V}_2)$ are solutions to (2). This completes the proof of the theorem. ♠

4   A gas-liquid interaction model

Consider the gas-liquid interaction model where a dissolved gas A and a dissolved reactant B interact in a bounded diffusion medium ω. The chemical reaction scheme is $A + \kappa_1 B \to \kappa_2 P$ and is called a second order reaction, where κ_1 and κ_2 are the rate constants and P is the product. Denoting by $u_1(x)$ and $u_2(x)$ the concentrations of the dissolved gas A and the reactant B, respectively, the above reaction scheme is governed by (1) with $L_i u_i = D_i \Delta u_i$, $f_i = \sigma_i u_1 u_2$, $i = 1, 2$, where $D_i > 0$, $i = 1, 2$, are the diffusion coefficients, and σ_1 is the rate constant, $\sigma_2 = \kappa_1 \sigma_1$. In a more general reaction scheme called the (l, s)th order reaction, the resulting equations are given by (1) with

$$f_i(x, u_1, u_2) = \sigma_i u_1^l u_2^s - q_i(x)\,, \quad i = 1, 2\,, \tag{18}$$

where $l \geq 1$, $s \geq 1$ are constants, $\sigma_i > 0$, $i = 1, 2$, are the rate constants and $q_i(x) \geq 0$, $i = 1, 2$, are possible internal sources [4]. The system (1) reduces to

$$- D_i \Delta u_i + \sigma_i u_1^l u_2^s - q_i(x) = 0, \quad x \in \omega, \quad u_i = g_i \geq 0 \text{ on } \partial\omega, \quad i = 1, 2.$$
(19)

The functions f_1 and f_2 are quasi-monotone nonincreasing in $(\mathbb{R}^k)^+ \times (\mathbb{R}^k)^+$.

The nonlinear difference scheme (2) reduces to

$$\mathcal{L}_i^h U_i(p) + f_i(p, U_1, U_2) = 0, \quad p \in \omega^h, \quad U_i = g_i \text{ on } \partial\omega^h, \quad i = 1, 2, \quad (20)$$

where f_i, $i = 1, 2$, are defined by (18) and $\mathcal{L}_i^h$, $i = 1, 2$, are approximations of $D_i \Delta$, $i = 1, 2$, on a mesh $\bar{\omega}^h$. We assume that the coefficients of the difference operators $\mathcal{L}_i^h$, $i = 1, 2$, satisfy (3).

Consider the mesh functions $(\widetilde{U}_1, \widehat{U}_2) = (W_1, 0)$ and $(\widehat{U}_1, \widetilde{U}_2) = (0, W_2)$, where W_1 and W_2 solve the problems

$$\mathcal{L}_i^h W_i(p) = q_i(p), \quad p \in \omega^h, \quad W_i(p) = g_i(p), \quad p \in \partial\omega^h, \quad i = 1, 2. \quad (21)$$

From here, (20) and taking into account that $f_i = 0$, $i = 1, 2$, on $(W_1, 0)$ and $(0, W_2)$, we conclude that $(\widetilde{U}_1, \widehat{U}_2) = (W_1, 0)$ and $(\widehat{U}_1, \widetilde{U}_2) = (0, W_2)$ satisfy conditions (8). By Lemma 1, from (21) we conclude that $W_i \geq 0$, $i = 1, 2$. From the above construction, we conclude that

$$\widetilde{\mathbf{U}}(p) = (W_1(p), W_2(p)), \quad \widehat{\mathbf{U}}(p) = (0, 0), \quad p \in \bar{\omega}^h, \quad (22)$$

are ordered upper and lower solutions of (2).

From (19), we conclude that in $\{\widehat{\mathbf{U}}(p) \leq \mathbf{U}(p) \leq \widetilde{\mathbf{U}}(p)\}$

$$\partial f_1(p, U_1, U_2)/\partial u_1 = \sigma_1 l U_1^{l-1}(p) U_2^s(p) \leq \sigma_1 l W_1^{l-1}(p) W_2^s(p), \quad p \in \bar{\omega}^h,$$
$$\partial f_2(p, U_1, U_2)/\partial u_2 = \sigma_2 s U_1^l(p) U_2^{s-1}(p) \leq \sigma_2 s W_1^l(p) W_2^{s-1}(p), \quad p \in \bar{\omega}^h.$$

From here, it follows that the assumptions in (9) hold true with

$$\bar{c}_1(p) = \sigma_1 l W_1^{l-1}(p) W_2^s(p) \,, \quad \bar{c}_2(p) = \sigma_2 s W_1^l(p) W_2^{s-1}(p). \qquad (23)$$

Thus, the monotone domain decomposition algorithm (4)–(6) with the above constructed ordered upper and lower solutions and the coefficients $\bar{c}_i$, $i = 1, 2$, satisfies Theorem 4 for solving the gas-liquid interaction model (19).

4.1   Numerical example

Consider the gas-liquid system (19) in the case of the second order reaction with $l = s = 1$ and zero internal sources. From (19), we write down the two-dimensional test problem in the non-dimensional variables

$$-\left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} \right) + \sigma_1 u_1 u_2 = 0 \,, \quad (x_1, x_2) \in \omega \,, \quad u_1 = 1 \text{ on } \partial\omega,$$

$$-\mu^2 \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \right) + \sigma_2 u_1 u_2 = 0 \,, \quad (x_1, x_2) \in \omega \,, \quad u_2 = 1 \text{ on } \partial\omega,$$

$$\omega = \{ 0 < x_1 < 1, 0 < x_2 < 1 \} \,, \quad \mu^2 = \frac{D_2}{D_1},$$

where u_1 and u_2 are the non-dimensional concentrations of the dissolved gas A and the reactant B, respectively. Since $\mu^2 \ll 1$, the test problem is singularly perturbed and u_2 has boundary layers, (that is, regions with rapid change of the solution) near $\partial\omega$ of width $\mathcal{O}(\mu)$.

For the differential operators, we use the central difference approximations on the five point rectangular stencil, which satisfy (3). We employ a layer adapted mesh of a piecewise uniform type. The piecewise uniform mesh is formed by the following manner. We divide each of the intervals $\bar{\omega}^{x_1} = [0, 1]$ and $\bar{\omega}^{x_2} = [0, 1]$ into three parts each $[0, \tau]$, $[\tau, 1-\tau]$ and $[1-\tau, 1]$. Assuming that $N_1 = N_2 = N$ and N is divisible by 4, in the parts $[0, \tau]$, $[1-\tau, 1]$ we

use the uniform mesh with $N/4 + 1$ mesh points, and in the part $[\tau, 1 - \tau]$ with $N/2 + 1$ mesh points. This defines the piecewise uniform mesh in the x_1- and x_2-directions. The transition points τ and $1 - \tau$ are determined by

$$\tau = \min\{4^{-1}, \mu \ln N\}.$$

If the parameter μ is small enough, then the uniform mesh inside of the boundary layers with the step size $h_\mu = 4\tau N^{-1}$ is fine, and the uniform mesh outside of the boundary layers with the step size $h = 2(1 - 2\tau)N^{-1}$ is coarse. The central difference schemes on the piecewise uniform mesh converge μ-uniformly to the solution of the test problem.

From (21)–(23), we conclude that

$$\widetilde{\mathbf{U}}(p) = (1,1)\,, \quad \widehat{\mathbf{U}}(p) = (0,0)\,, \quad p \in \bar{\omega}^h\,, \quad \bar{c}_i = \sigma_i\,, \quad i = 1,2.$$

We choose $\sigma_1 = 2$, $\sigma_2 = 1$ and $\delta = 10^{-5}$ in the stopping criterion. We use balanced domain decompositions, where the mesh points are equally distributed among the main subdomains, and solve all linear systems in the monotone domain decomposition algorithm (4)–(6) by ICCG-method. The initial guess is

$$\bar{V}_1^{(0)}(p) = 1\,, \quad p \in \bar{\omega}^h\,, \quad \underline{V}_2^{(0)}(p) = 0\,, \quad p \in \bar{\omega}^h.$$

Our numerical results show that for all values of μ and N and all domain decompositions, the sequence $\{\bar{V}_1^{(n)}, \underline{V}_2^{(n)}\}$ generated by (4)–(6) converges monotonically to the exact solution of the nonlinear difference scheme. This confirms the theoretical results in Theorem 4.

For $\mu \leq 10^{-2}$, the convergence iteration count for the monotone undecomposed algorithm with $M = 1$ is uniform with respect to μ and N and equals three. Our numerical results show, that if $\mu \leq 10^{-2}$, then for N and M fixed, the number of iterations n_0, required to reach the prescribed accuracy δ, is independent of μ. Table 1, for various values of N and M, gives

TABLE 1: Number of iterations n_0 for algorithm (4)–(6).

$M \backslash N$	32	64	126	256	512
4	3	3	3	3	3
8	7	6	5	5	4
16	21	18	15	13	11
32	n.a.	53	44	36	28

the number of iterations n_0 for the monotone domain decomposition algorithm (4)–(6). From Table 1, we conclude that for M fixed, the number of iterations is a monotone decreasing function with respect to the number of mesh points N. For $M \leq 8$, the numbers of iterations for the monotone domain decomposition algorithm (4)–(6) are almost the same as for the monotone undecomposed algorithm. The last numerical results are important in the context of parallel implementation of the monotone domain decomposition algorithm (4)–(6).

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