Joint pricing and lot sizing models with discount: A geometric programming approach

M. Esmaeili[1] P. Zeephongsekul[2]
Mir-Bahador Aryanezhad[3]

(Received 27 July 2007; revised 26 October 2007)

Abstract

We propose a novel joint pricing and lot sizing model to enable manufacturers plan production and pricing. These types of models have proven to be very popular and are collectively known as the Joint Pricing and Lot sizing Models. We include a discount factor in our model to increase profit for the manufacturer. Our proposed model relies on the fact that demand influences production cost indirectly, while it is dependent on price and the discount offered. By considering the form of demand and production cost, it is apparent that the presented model is a *Signomial* Geometric Programming problem. We obtain optimal solutions for price, lot size and discount factor by applying the modified transformation method of geometric programming. Numerical examples, which include sensitivity analysis of the objective function and parameters, illustrate our model.

See http://anziamj.austms.org.au/ojs/index.php/ANZIAMJ/article/view/323 for this article, Published October 28, 2007. ISSN 1446-8735

1 Introduction

Since pricing and lot sizing decisions are entwined in the production environment, models which are based on jointly determining optimal values for prices and lot size are extremely popular and they are collectively known as Joint Pricing and Lot sizing Models (JPLM). These models are also practical and efficient because emphasis is placed simultaneously on production and market demand. In contrast with classical EPQ (Economic Production Quantity), in JPLM demand is not fixed and could depend on price and production cost. Several researchers have considered JPLM where price depends on demand over a planning horizon [2, 9, 11, 8]. They used the primal and dual of Geometric Programming (GP) problems to determine the optimal price and lot size. Beightler [3] and Duffin et al. [6] provide an extensive discussion on the primal and dual (GP) problems. Many researchers also considered the effects of different parameters on demand and cost in JPLM, for example, Freeland [7], Sajadi et al. [12] and Lee and Kim [11] considered the impact of marketing expenditure on demand in addition to price, whereas Lee et al. [8] considered the impact of reliability on demand in addition to price.

Quantity discount, as a popular tool of coordination mechanism, has also been considered in supply chain work, notably by Chiang et al. [4]. Similar approaches have been used by Corbett and de Groote [5], Viswanathan & Wang [14] and Weng [15]. Note that the demand in these papers is unaffected by the size of the discount and, when it is price sensitive, there is opportunity for the manufacturer to decrease production costs and to increase revenue. Abad [1] and Lee [10] included the quantity discount in their model when demand is price sensitive. Quantity discount is not in itself sufficiently effective without considering volume discount [14]. Therefore, here we assume that the supplier gives quantity discount to a manufacturer while the manufacturer also offers volume discount on the product to customers. The quantity discount is primarily offered with the intention of decreasing production cost when production volume increases, thereby achieving reduction in production cost. On the other hand the demand imposed on the manufacturer is sensitive to its price and discount, therefore demand increases by increasing the size of the discount. This approach leads to more benefit for the manufacturer.

In this article, unlike most models cited above, we present a JPLM where volume of production is allowed to be different from demand, which makes it more realistic and distinguishes it from other models. We also consider production, discount, setup and holding costs in our model. The holding cost subsumes the costs associated with investing in inventory and maintaining the physical investment in storage. The holding cost incorporates items such as capital costs, taxes, insurance, handling, storage, shrinkage, obsolescence, and deterioration. In our model, the holding cost is expressed in term of a percentage of production cost [13, e.g.]. The optimal decision variables, that is price, lot size, production volume and discount are obtained in closed form.

2 Notation and problem formulation

This section introduces the notation and formulation of our model. Here, we state decision variables and input parameters, see Table 1, and assumptions underlying our models.

2.1 Assumptions

Our proposed model is based on the following assumptions:

1. Parameters are deterministic and known in advance;
2. Shortages are not permitted;
3. The model is profitable;

Table 1: decision variables and input parameters.

X	Production volume (decision variable)
P	price (decision variable)
d	Discount (decision variable)
Q	Lot size (decision variable)
A	Setup cost (\$/setup)
C	The production cost per unit
k	Scaling constant for demand ($k > 0$)
i	Percentage of the inventory holding cost per unit of production cost
α	Price elasticity of demand ($\alpha > 1$)
u	Scaling constant for production cost ($u > 0$)
β	Production volume elasticity of demand ($0 < \beta < 1$)
μ	Discount elasticity of demand ($0 < \mu < 1\,,\ \beta(\alpha - \mu) < 1\,,\ \alpha - \mu > 2$)
$D(P, d)$	Demand; for notational simplicity we let $D \equiv D(P, d)$

4. The demand is a function of price P which is similar to the demand function introduced by Kim and Lee [9], this function depends on the discount d according to

$$D(P, d) = kP^{-\alpha} d^{\mu}\,; \tag{1}$$

5. The supplier offers quantity discount to the manufacturer and the production cost C is a function of production volume X according to

$$C = uX^{-\beta}\,. \tag{2}$$

3 The proposed model

In the proposed model, the net profit obtained by a manufacturer is a function of production volume, lot size, price and discount. The manufacturer's objective function is similar to the objective function proposed by Abad [2] and Lee [10]. However, we include a discount component d and production volume X as new decision variables. The manufacturer's objective function is

$$\begin{aligned}\text{Manufacturer's Profit} &= \text{Sales Revenue} - \text{Production Cost}\\ &- \text{Discount Cost} - \text{Setup Cost} - \text{Holding Cost}.\end{aligned}$$

The above is expressed as a function of the decision variables as

$$\begin{aligned}\Pi(X,Q,P,d) &= PX - CX - dX - AXQ^{-1} - 1/2iCQ\,,\\ \text{subject to}\quad X &\leq D = kP^{-\alpha}d^{\mu}\,, \\ 0 &\leq X,Q,P,d\,.\end{aligned} \tag{3}$$

The constraint of the model indicates that production volume is not necessarily equal to demand, as is commonly assumed. Note that the holding cost which appears in (3) is the standard one used in inventory control models [13, e.g.]. The holding cost is expressed as a percentage of production cost associated with the lot size, that is iCQ. This is further multiplied by $1/2$ to obtain an average value due to the fact that lot size is declining uniformly because of continuous demand. The optimal solution is obtained in the next section.

3.1 The optimal solution

The proposed model is a *Signomial* problem in Geometric Programming (GP) with one degree of difficulty. Since standard GP fails to guarantee global optimality for any *Signomial* GP, we apply the method used by Lee and Kim [9] to transform the *Signomial* GP problem into a *Posynomial* GP problem. They assumed that there is a lower bound Π_0 for the objective function such that maximizing Π_0 is equivalent to maximizing the objective function. Therefore, the *Signomial* GP is transformed into a *Posynomial* GP with an additional constraint and variable Π_0:

$$\begin{aligned} \max \Pi_0 &= \min \Pi_0^{-1}\,, \\ \text{subject to} \quad 1 &\geq k^{-1} P^{\alpha} d^{-\mu} X\,, \\ 1 &\geq u X^{-\beta} P^{-1} + A P^{-1} Q^{-1} + d P^{-1} \\ &\quad + 0.5 i u P^{-1} Q X^{-(\beta+1)} + \Pi_0 P^{-1} X^{-1}\,, \\ 0 &\leq \Pi_0, X, Q, P, d\,. \end{aligned} \tag{4}$$

This *Posynomial* problem can be solved by its dual objective function. Appendix A provides details of the dual formulation and solution procedure.

Let λ, w_{ij}, $i = 1, \ldots, 5$ and $j = 1, 2$ be the dual variables and define

$$\delta_{ij} = \frac{w_{ij}}{\lambda}\,. \tag{5}$$

Note that $\delta_{ij}s$ are the weights of the terms in the constraints (4) and they are obtained in (22) of Appendix A where $\sum_{i=1}^{5} \delta_{i2} = 1$ and $\sum_{i=1}^{5} w_{i2} = \lambda$.

The weights represent proportions of production cost δ_{12}, setup cost δ_{22}, discount cost δ_{32}, inventory holding cost δ_{42} and profit δ_{52} to the total sales revenue. The following relations hold:

$$\delta_{11} = k^{-1} P^{\alpha} d^{-\mu} X\,, \tag{6}$$

$$\delta_{12} = u P^{-1} X^{-\beta}\,, \tag{7}$$

$$\delta_{22} = A P^{-1} Q^{-1}\,, \tag{8}$$

$$\delta_{32} = P^{-1} d\,, \tag{9}$$

$$\delta_{42} = 0.5 i u P^{-1} Q X^{-\beta-1}\,, \tag{10}$$

$$\delta_{52} = P^{-1} X^{-1} \Pi_0\,. \tag{11}$$

Solving (6) to (11) for the optimal solutions, we easily obtain

$$P^* = (k u^{-1/\beta} \delta_{32}^{\mu} \delta_{12}^{1/\beta} X^{-\beta})^{1/\alpha - \mu - 1/\beta}\,, \tag{12}$$

$$Q^* = A\delta_{22}^{-1} P^{-1}\,, \tag{13}$$
$$d^* = \delta_{32} P\,, \tag{14}$$
$$X^* = (u\delta_{12}^{-1} P^{-1})^{1/\beta}\,, \tag{15}$$
$$\Pi_0^* = \delta_{52} P X\,. \tag{16}$$

4 Sensitivity analysis

Manufacturers need to understand how varying key parameters affect the optimal solutions, and where this helps them to improve their current policy. In our model, sensitivity analysis estimates the effects of under/over estimation of parameters on the optimal solution using the relationships between parameters and variables. Our findings are listed below.

1. The estimates of the weights of the terms in the constraints (δ_{ij}) is determined from the input parameters. For example, if the proportion of holding cost to sales revenue δ_{42} decreases, then proportion of production cost to the sales revenue δ_{12} and proportion of profit to the sales revenue δ_{52} will increase. Furthermore, the proportion of setup cost to sales revenue δ_{22} will decrease. Also, if discount quantity elasticity μ increases or price elasticity α decreases then proportion of discount cost to the sales revenue, δ_{32}, will increases.

   Proof: These assertions are directly deduced from the following equations obtained from (5) and (23)–(28) in Appendix A:

$$\delta_{12} - \delta_{52} = \left(((\beta+1)(\alpha-\mu)-2)\right)\alpha^{-1}(1-\beta)^{-1}\,, \tag{17}$$
$$\delta_{42} + \delta_{12} = (\alpha-\mu-1)\alpha^{-1}(1-\beta)^{-1}\,, \tag{18}$$
$$\delta_{42} + \delta_{52} = (1+\beta(-\alpha+\mu))\alpha^{-1}(1-\beta)^{-1}\,, \tag{19}$$
$$\delta_{32} = \mu\alpha^{-1}\,, \tag{20}$$

$$\delta_{22} = \delta_{42}\,. \tag{21}$$

♠

2. From the formula for R given in (37) of Appendix B, it follows that R would increase if A, u or i increases. R also increases if k decreases.

   Proof: This follows directly from (30) in Appendix B. ♠

3. If $R > 0$, proportions of set up cost, δ_{22}, and holding cost to sales revenue, δ_{42}, will increase and proportion of production cost to the sales revenue, δ_{12}, will decrease.

   Proof: Let $\delta_{ij}^* = \delta_{ij} + \partial\delta_{ij}$. Using (5) and (23)–(28) we have

$$\begin{aligned} \delta_{42}^* - \delta_{42} &= \frac{w_{42} + \partial w_{42}}{\lambda + \partial\lambda} - \frac{w_{42}}{\lambda} \\ &= \frac{\alpha J^{-1} R}{\lambda(\lambda + \partial\lambda)}\,. \end{aligned}$$

   If $R > 0$, then $\delta_{42}^* - \delta_{42}$ is positive which means $\delta_{42}^* > \delta_{42}$. Therefore, based on finding 1, we obtain $\delta_{22}^* > \delta_{22}$ and $\delta_{12} < \delta_{12}^*$. ♠

4. If the proportion of set up cost to the sales revenue δ_{22} decreases then price P and discount d will increase and production volume X and lot size Q will decrease.

   Proof: These assertions are direct results from (12) to (15). ♠

5 Numerical example

This section illustrates our model by presenting an example including a sensitivity analysis of the optimal solution. Consider a manufacturer supplying goods which are elastic ($\alpha > 1$) with many substitutes in a very competitive market. An example of this model would be a factory which supply parts for a petrochemical industry. Let the production scenario be as follows. The holding cost constitute 50% of production cost where the production cost is a function of production volume and is $C = 0.2X^{-0.1}$. The set up cost is set at 1.8 and the supplier offers 0.1 discount per production unit. In addition, the demand function is set to $D(P, d) = 5P^{-2.3}d^{0.2}$. The manufacturer would like to determine an optimal policy on lot size, production volume, selling price and discount. We solve the proposed model by following the procedure in Appendix A, which results in $w_{11}^* = 3.1$. The other optimal weights are $(w_{12}^*, \ldots, w_{52}^*) = (2.1, 1.7, 0.6, 1.7, 1)$, $\lambda^* = 7.1$, $(\delta_{11}^*, \delta_{12}^*, \delta_{22}^*, \delta_{32}^*, \delta_{42}^*, \delta_{52}^*) = (1, 0.3, 0.2, 0.1, 0.2, 0.1)$. Using the weights, the lower bound, Π_0, is 0.7, the maximum profit of the manufacturer is 1.6 unit, the selling price is \$0.6 per unit, the production volume is 8 units per week with 13 units lot sizes per two weeks, and the discount is \$0.05. Limitations on production equipments makes the manufacturer produce only 8 units per week whereas his lot size is 13 for almost two working weeks. Even though the manufacturer can choose the lot size equal to the production volume, in an optimal solution the manufacturer obtains more profit ($1.7 > 1.1$) by choosing a lot size greater than the production volume.

Consider the example when δ_{12} is underestimated to 0.15 instead of 0.3. From (17) to (19) we have $(\delta_{22}, \delta_{42}, \delta_{52}) = (0.38, 0.38, 0.00024)$, and consequently applying (12) to (15) $(P^*, Q^*, X^*, d^*) = (1.3, 3.8, 1.8, 0.1)$. Observe that an underestimated δ_{12} results in a higher price and discount and lower lot size and production volume.

Suppose the setup cost increases to two and the other parameters do not change. A manufacturer is interested to know what is the effect of this parameter on the optimal solution. From (30) to (37) we obtain $J = 0.61$, $\Delta A/A = 0.11$, $R = 0.097$, $(w_{12}, w_{22}, w_{32}, w_{42}) = (2.2, 1.8, 0.6, 1.8)$, $\lambda = 7.5$ and the corresponding estimated solution is $(P^*, Q^*, X^*, d^*) = (0.6, 14.5, 8.2, 0.05)$. The policy implication is that the price and discount should remain almost the same but the lot size should be increased by about 1.1 units and production volume should be decreased by about 0.3 units. This new suggested policy is essentially optimal without solving the problem again.

The effects of changes in parameters on the optimal solution are also seen in the given example. Assume inventory holding cost rate decreases to 0.1 instead of 0.5. According to finding 4 we expect price and discount to decrease and production volume and lot size to increase. This is achieved using (12) to (15) from which we obtain $(P^*, Q^*, X^*, d^*) = (0.4, 34.4, 17.4, 0.03)$.

6 Conclusion

Discount in JPLM is an effective tool to increase manufacturer's profit. We made two assumptions: demand is affected by the size of discount, and production cost decreases when production volume increases. Our proposed model is more realistic than the state-of-the-art models since market demand is related to production volume. It is a *Signomial* problem and was solved using standard Geometric Programming. We also performed sensitivity analysis to investigate the effect of changes of parameters on the optimal solution.

There is much scope to extend the present work. For example, parameters and decision variables can be considered random or even fuzzy. Other parameters of a distributed system which were not included in this article, such as production rate or shortage cost, could be added to the model. Finally, the proposed model can be made more realistic by extending it to cases where multiple products are being sold in several markets.

A The dual

The model described in (3) is a *Posynomial* GP problem with one of degree of difficulty. Its dual is

$$
\begin{aligned}
\max F(w_{11}) &= \left[\frac{1}{w_0}\right]^{w_0} \left[\frac{k^{-1}}{w_{11}}\right]^{w_{11}} \left[\frac{u}{w_{12}}\right]^{w_{12}} \left[\frac{A}{w_{22}}\right]^{w_{22}} \\
&\quad \times \left[\frac{1}{w_{32}}\right]^{w_{32}} \left[\frac{0.5iu}{w_{42}}\right]^{w_{42}} \left[\frac{1}{w_{52}}\right]^{w_{52}} \lambda^{\lambda}, \\
\text{subject to} \quad & w_0 = 1\,, \\
& -w_0 + w_{52} = 0\,, \\
& \alpha w_{11} - w_{12} - w_{22} - w_{32} - w_{42} - w_{52} = 0\,, \\
& -\mu w_{11} + w_{32} = 0\,, \\
& w_{11} - \beta w_{12} - (\beta+1) w_{42} - w_{52} = 0\,, \\
& w_{22} - w_{42} = 0\,.
\end{aligned} \tag{22}
$$

The first constraint is the Normality Condition and the others are the Orthogonally Conditions [3, 6]. When the dual variables are rewritten in terms of one dual variable, w_{11}, the following equations resulted:

$$w_0 = 1\,, \tag{23}$$

$$w_{12} = \left[((\beta+1)(\alpha-\mu)-2)(1-\beta)^{-1} w_{11}\right] + 1\,, \tag{24}$$

$$w_{22} = w_{42} = \left[(-\beta\alpha + \beta\mu + 1)(1-\beta)^{-1} w_{11}\right] - 1\,, \tag{25}$$

$$w_{32} = \mu w_{11}\,, \tag{26}$$

$$w_{52} = 1\,, \tag{27}$$

$$\lambda = \alpha w_{11}\,. \tag{28}$$

In order for the dual to have a feasible solution, we make the additional assumptions that $\beta(\alpha-\mu) < 1$ and $\alpha - \mu > 2$. By substituting the dual variables from (23) to (28) into $F(w_{11})$ and then taking the logarithm, we obtain a concave programming problem:

$$\max \log F(w_{11}) = -w_{11}\log kw_{11} - w_{12}\log w_{12}u^{-1} - w_{22}\log w_{22}A^{-1} - w_{32}\log w_{32} - w_{22}\log w_{22}(0.5iu)^{-1} + \lambda\log\lambda\,. \quad (29)$$

Note that (29) can be shown to be a concave function over all values of $w_{i2} > 0$, $i = 1, \ldots, 5$, $\sum_{i=1}^{5} w_{i2} = \lambda$. Such a problem can be solved by any line search technique [12]. According to the duality theory of GP, $F^* = 1/\max\Pi$ and this is then used to obtain the optimal solution from (12) to (15).

B Dual sensitivity

Here we evaluate the effects of percentage changes in input parameters (excluding α, β and μ) on the dual variables. This approach employed is similar to Lee [10]. The effects are calculated as follows:

$$\partial w_{11} = J^{-1}R\,, \quad (30)$$

$$\partial w_{12} = \left[((\beta+1)(\alpha-\mu)-2)(1-\beta)^{-1}\right] J^{-1}R\,, \quad (31)$$

$$\partial w_{22} = \partial w_{42} = \left[(-\beta\alpha+\beta\mu+1)(1-\beta)^{-1}\right] J^{-1}R\,, \quad (32)$$

$$\partial w_{32} = \mu J^{-1}R\,, \quad (33)$$

$$\partial w_{52} = \partial w_0 = 0\,, \quad (34)$$

$$\partial\lambda = \alpha J^{-1}R\,, \quad (35)$$

$$J = \frac{1}{w_{11}^*} + \frac{\left[\frac{((\beta+1)(\alpha-\mu)-2)}{(1-\beta)}\right]^2}{w_{12}^*} + \frac{\left[\frac{(-\beta\alpha+\beta\mu+1)}{(1-\beta)}\right]^2}{w_{22}^*} + \frac{\mu^2}{w_{32}^*} + \frac{\left[\frac{(-\beta\alpha+\beta\mu+1)}{(1-\beta)}\right]^2}{w_{42}^*} - \frac{\alpha^2}{\lambda^*}\,, \quad (36)$$

$$R = \frac{\Delta k^{-1}}{k^{-1}} + \left[\frac{((\beta+1)(\alpha-\mu)-2)\Delta u}{(1-\beta)u}\right] + \left[\frac{(-\beta\alpha+\beta\mu+1)\Delta A}{(1-\beta)A}\right]$$

$$+\left[\frac{(-\beta\alpha+\beta\mu+1)\Delta 0.5iu}{(1-\beta)0.5iu}\right]. \tag{37}$$

J is the Jacobian of the dual objective function $F(w_{11})$ and $\Delta R/R$ represents the percentage change in a primal problem coefficient R. Note that if w_0, w_{52} are fixed, then $\partial w_0, \partial w_{52}$ will equal zero. Therefore, we readily compute the new dual variables $\lambda = \lambda^* + \partial\lambda$, $w_{ij} = w_{ij}^* + \partial w_{ij}$ and then use these to calculate the weights, δ_{ij}, and the associated primal variable values.

References

[1] Prakash L. Abad. Determining optimal selling price and the lot size when the supplier offers all-unit quantity discounts. *Decision Sciences*, 3(19):622–634, 1988. C140

[2] Prakash L. Abad. Supplier pricing and lot sizing when demand is price sensitive. *European Journal of Operational Research*, (78):334–354, 1994. http://cat.inist.fr/?aModele=afficheN&cpsidt=3309143. C140, C143

[3] Charles S. Beightler. *Applied Geometric Programming*. John Wiely and Sons, 1976. C140, C149

[4] C. W. Chiang, J. Fitzsimmons, Z. Huang, and S. Li. A game theoretic approach to quantity discount problem. *Decision Sciences*, 25(1):153–168, 1994. doi:10.1111/j.1540-5915.1994.tb00521.x. C140

[5] C. J. Corbett and X. de Groote. A supplier's optimal quantity discount policy under asymmetric information. *Management Science*, 46(3):444–450, 2000. C140

[6] R. J Duffin, E. L Peterson, and C. Zener. *Geometric programming, Theory and Application*. John Wiely and Sons, 1967. C140, C149

[7] J. R. Freeland. Coordination strategies for production and marketing in a functionally decentralized firm. *AIIE Transactions*, 12:126–132, 1982. C140

[8] Woon J.Lee, Daesoo Kim, and A. Cabot. Optimal demand rate, lot sizing, and process reliability improvement decisions. *IIE Transactions*, 28:941–952, 1996. `http://cat.inist.fr/?aModele=afficheN&cpsidt=2490704`. C140

[9] Daesoo Kim and Woon J. Lee. Optimal joint pricing and lot sizing with fixed and variable capacity. *European Journal of Operational Research*, 109(1):212–227, 1998. `http://www.sciencedirect.com/science/article/B6VCT-3TMR6M6-H/2/2d166a9b5fc0a9a27e923c495f256663`. C140, C142, C143

[10] Woon J. Lee. Determining order quantity and selling price by geometric programming. *Decision Sciences*, 24:76–87, 1993. doi:10.1111/j.1540-5915.1993.tb00463.x. C140, C143, C150

[11] Woon J. Lee and Daesoo Kim. Optimal and heuristic decision strategies for integrated product and marketing planning. *Decision Sciences*, 24(6):1203–1213, 1993, doi:10.1111/j.1540-5915.1993.tb00511.x. C140

[12] Seyed J. Sajadi, Maryam Orouge, and M. B. Aryanezhad. Optimal production and marketing planning. *Computational Optimization and Applications*, 30(2):195–203, 2005. `http://portal.acm.org/citation.cfm?id=1062644.1062668`. C140, C150

[13] Richard J. Tersine. *Principles of Inventory And Materials Management*. PTR Prentice Hall, Englewood Cliffs, New Jersey 07632, 1994. C141, C143

[14] S. Viswanathan and Q.Wang. Discount pricing decisions in distribution channels with price sensitive demand. *European Journal of*

Operational Research, 149(3):571–587, 2003. http://ideas.repec.org/a/eee/ejores/v149y2003i3p571-587.html. C140

[15] Z. Kevin Weng. Channel coordination and quantity discounts. *Management Sciences*, 41(9):1509–1522, 1995. http://www.jstor.org/view/00251909/di012962/01p03025/0. C140

Author addresses

1. **M. Esmaeili**, Dept. Industrial Engineering, Iran University of Science and Technology, Tehran, Iran.
mailto:maryam.esmaeili@rmit.edu.au

2. **P. Zeephongsekul**, School of Mathematical and Geospatial Sciences, RMIT University, Melbourne, Victoria,Australia.

3. **Mir-Bahador Aryanezhad**, Dept. Industrial Engineering, Iran University of Science and Technology, Tehran, Iran.