Differential games with many pursuers when evader moves on the surface of a cylinder

A. Sh. Kuchkarov[1] M. H. Risman[2] A. H. Malik[3]

(Received 28 June 2010; revised 9 December 2011)

Abstract

We study a pursuit differential game with many Pursuers when the Evader moves on the surface of a given cylinder. Maximal speeds of all players are equal. We consider two cases: in the first case, the Pursuers move arbitrarily without phase constraints; and in the second case, the Pursuers move on the surface of the cylinder. In both cases, we give necessary and sufficient conditions to complete the pursuit. In addition, in the second case, we show that pursuit differential game on a cylinder are equivalent to a differential game on the plane with many groups of Pursuers where each group consists of infinite number of pursuers having the same control parameter.

---

http://journal.austms.org.au/ojs/index.php/ANZIAMJ/article/view/3280 gives this article, Published January 12, 2012. ISSN 1446-8735.

Contents

1 Introduction

Simple motion differential games are studied in many papers. Fundamental results are obtained by Isaacs [1], Petrov [2], Petrosyan [3, 4], Pshenichnii [5], and Chernous'ko [6].

The games with phase constraints are of special interest in studying the simple motion differential games.

Simple motion pursuit games with many Pursuers are investigated by Ivanov [7] under the assumption that all players move in a convex set and have equal maximal speed. In the works of Melikyan and Ovakimyan [8, 9], Kuchkarov [10, 11] differential games are studied on the Riemannian Manifold. Azamov [12] examined the evasion problem in the case when the evader moves along the given curve and its maximal speed is greater than that of the Pursuer. In the work of Kuchkarov and Rikhsiev [13], a pursuit differential games are investigated when all players have identical maximal speed and the Evader moves along a strictly convex smooth hypersurface. In the work of Ibragimov [14] a pursuit-evasion differential game is studied in a convex compact set when control functions of players are subject to integral constraints.

We study a pursuit differential game with many Pursuers when the Evader moves on the surface of a given cylinder. Maximal speeds of all players are equal. We consider two cases: in the first case, the Pursuers move arbitrarily without phase constraints and in the second case, Pursuers move on the surface of the cylinder. The Pursuers use a counter strategy and the Evader use a positional strategy. The trajectories of the Pursuers are defined as the absolutely continuous solutions of the differential equations, the trajectory of the Evader is defined as stepwise movement. In both cases, we give necessary and sufficient conditions to complete the pursuit. In addition, in the second case we show that pursuit differential games on a cylinder are equivalent to a differential game on the plane with many groups of Pursuers where each group consists of an infinite number of Pursuers having the same control parameter.

2 Statement of the Problem

Let

$$M = \left\{ x = (x_{(1)}, x_{(2)}, x_{(3)}) \in \mathbb{R}^3 \mid x_{(1)}^2 + x_{(2)}^2 = R^2,\ x_{(3)} \in \mathbb{R} \right\}$$

(the sub indices in parentheses denote the coordinates of a point), and M_z, $z \in M$, be the tangent plane to the surface of the cylinder M at the point z.

The motions of the Pursuers P_i and the Evader E are described by

$$P_i : \dot{x}_i = u_i\,,\ x_i(0) = x_{i0}\,, \quad E : \dot{y} = v\,,\ y(0) = y_0\,, \tag{1}$$

where $x_i, y, u_i, v \in \mathbb{R}^3$; u_i and v are control parameters, $i = 1, 2, \ldots, m$.

Definition 1. *A measurable function $v(\cdot) = (v(t), t \geqslant 0)$ is called a control of the Evader E if $|v(t)| \leqslant 1$, $t \geqslant 0$, and the solution $y(\cdot) = (y(t), t \geqslant 0)$ of the initial value problem $\dot{y} = v(t)$, $y(0) = y_0$, satisfies the inclusion*

$$y(t) \in M\,, \quad t \geqslant 0\,. \tag{2}$$

Definition 2. *A measurable function $u_i(\cdot) = (u_i(t), t \geqslant 0)$ is called a control of the Pursuer P_i if $|u_i(t)| \leqslant 1$, $t \geqslant 0$, and the solution $x_i(\cdot) = (x_i(t), t \geqslant 0)$*

of the initial value problem $\dot{x}_i = u_i(t)$, $x_i = x_{i0}$ *satisfies the inclusion*

$$x_i(t) \in N\,, \quad t \geqslant 0\,, \tag{3}$$

where N *is a given set.*

Definition 3. *A function* $U_i : [0,\infty) \times N \times M \times B \to B$, $i = 1, 2, \ldots, m$, *is called a strategy of the Pursuer* P_i *if for any control* $v(\cdot)$ *of the Evader, the initial value problem*

$$\begin{cases} \dot{y} = v(t), & y(0) = y_0\,, \\ \dot{x}_i = U_i\left(t, x_i, y, v(t)\right), & x_i(0) = x_{i0}\,, \end{cases}$$

has a unique absolutely continuous solution $(x_i(\cdot), y(\cdot))$, *where the set* $B = \left\{z \in \mathbb{R}^3 \mid |z| \leqslant 1\right\}$ *and the function* $u_i(t) = \dot{x}_i(t)$ *is a control of the Pursuer* P_i. *The function* $x_i(\cdot)$ *is called the trajectory of the Pursuer* P_i *generated by the strategy* U_i, *the initial state* (x_{i0}, y_0) *and the control* $v(\cdot)$.

Definition 4. *We say that pursuit can be completed for the time* T *in game* (1)–(3) *if there exist strategies* $U_1, U_2, \ldots, U_m$ *of the Pursuers* $P_1, P_2, \ldots, P_m$, *respectively, such that for any control* $v(\cdot)$ *of the Evader* E *the trajectories satisfy the condition* $x_i(t) = y(t)$ *for some* $t \in (0, T]$ *and* $i \in \{1, 2, \ldots, m\}$.

Definition 5. *A function* $V : [0,\infty) \times N^m \times M \longrightarrow TM$ *is called a strategy of the Evader if for any* $(x_1, x_2, \ldots, x_m, y) \in N^m \times M$ *holds, then the inclusion* $V(t, x_1, x_2, \ldots, x_m, y) \in B_y$, *where* TM *is the tangent bundle of the cylinder* M *as a manifold and* B_y *is unit ball on the tangent plane* M_y *with centre at* $y \in M_y$.

Let $\Delta = \{0 = t_0, t_1, t_2, \ldots\}$ be a partition of the interval $[0, \infty)$. The trajectory $y(\cdot)$ of the Evader generated by Pursuer controls $u_1(\cdot), u_2(\cdot), \ldots, u_m(\cdot)$, and a strategy V, and an initial position $(x_{10}, x_{20}, \ldots, x_{m0}, y_0)$ is constructed:

1. at the time t_i, we determine the vector

$$v_i = V\left(t_i, x_1(t_i), x_2(t_i), \ldots, x_m(t_i), y(t_i)\right), \quad v_i \in B_{y(t_i)},$$

and construct a geodesic $\gamma_{yi} : [0; t_{i+1} - t_i] \to M$ for which

$$\gamma'_{yi}(0) = \nu_i\,, \quad \gamma_{yi}(0) = y(t_i), \quad \left|\gamma'_{yi}(s)\right| = 1\,, \quad s \in [0; t_{i+1} - t_i];$$

2. the trajectory $y(\cdot)$ is defined as the solution of the equation

$$\dot{y} = \gamma'_{yi}(t - t_i), \quad t \in [t_i; t_{i+1})\,, \quad i = 1, 2, \dots\,, \quad y(0) = y_0.$$

Definition 6. *We say that evasion is possible in game (1)–(3) if for the given initial position $(x_{10}, x_{20}, \dots, x_{m0}, y_0)$, $x_{i0} \neq y_0$, there exist a partition Δ of the interval $[0, \infty)$, and a strategy V of the Evader E such that for any Pursuers' controls $u_1(\cdot), u_2(\cdot), \dots, u_m(\cdot)$ the corresponding trajectories satisfy the inequalities $x_i(t) \neq y(t)$ for all $t \geqslant 0$ and $i \in \{1, 2, \dots, m\}$.*

We consider the following problems

Problem 1. Find a condition to complete the pursuit in the game (1)–(3) and construct the strategy for the Pursuer.

Problem 2. Find a condition for the evasion to be possible in the game (1)–(3) and construct the strategy for the Evader.

3 Main results

At the beginning part of this section we consider game (1)–(3) in the case when the Pursuers move without phase constraints: $N = \mathbb{R}^3$.

Auxiliary game on a half cylinder In the differential game (1)–(3) let one Pursuer exist (that is, $m = 1$) and the Pursuer moves without phase constraints (that is, $N = \mathbb{R}^3$) and the Evader moves on the surface

$M_+ = \{z \in M | z_{(3)} \geqslant 0\}$, particularly, $y_{0(3)} \geqslant 0$. M_+ is a half cylinder. For short, we use x and u instead of x_1 and u_1.

Theorem 7. *1. If $x_{0(3)} \leqslant y_{0(3)}$, then evasion is possible in the game (1)–(3).*

2. If $x_{0(3)} > y_{0(3)}$, then pursuit can be completed in the game (1)–(3) for a finite time $T(x_0, y_0)$.

Proof:

1. Let an initial points x_0, y_0 be given such that $0 \leqslant x_{0(3)} \leqslant y_{0(3)}$ and $x_0 \neq y_0$. If the Evader uses the control $v(t) = (0, 0, 1)$, $t \geqslant 0$, then according to property of obtuse triangle we have

$$|x_0 - y(t)|^2 \geqslant |x_0 - y_0|^2 + |y_0 - y(t)|^2 = |x_0 - y_0|^2 + t^2 > t^2.$$

   On the other hand, for any control $u(\cdot)$ of the Pursuer $|u(t)| \leqslant 1$ holds. Then for the trajectory $x(\cdot)$ we have $|x_0 - x(t)| \leqslant t$. Hence, by using the triangle inequality, we obtain

$$\begin{aligned} &|x(t) - y(t)| \geqslant |y(t) - x_0| - |x(t) - x_0| > t - t = 0, \quad t \geqslant 0, \\ &\Rightarrow x(t) \neq y(t), \quad t \geqslant 0, \end{aligned}$$

   that is evasion is possible.

2. Now, we turn to the case $x_{0(3)} > y_{0(3)} \geqslant 0$ and prove that pursuit can be completed in the game (1)–(3), for some time $T(x_0, y_0)$. Let

$$z_0 = \left(0, 0, z_{0(3)}\right), \quad z_{0(3)} = \frac{|x_0|^2 - y_{0(3)}^2}{x_{0(3)} - y_{0(3)}}, \quad t_1 = |x_0 - z_0|.$$

   We construct a strategy on an interval for the Pursuer $[0, t_1)$ to guarantee the strict inequality $x_{(3)}(t_1) > y_{(3)}(t_1)$ and equalities $x_{(j)}(t_1) = 0$, $j = 1, 2$. Then

$$|x_0 - z_0| < z_{0(3)} - y_{0(3)}. \tag{4}$$

We construct the Pursuer's strategy on $[0, t_1)$ as

$$U(t, x, y, v) = U_0 = \frac{z_0 - x_0}{|z_0 - x_0|}.$$

Thus

$$x(t_1) = x_0 + \int_0^{t_1} U_0 dt = x_0 + \int_0^{t_1} \frac{z_0 - x_0}{|z_0 - x_0|} dt = x_0 + \frac{z_0 - x_0}{|z_0 - x_0|} t_1 = z_0 .$$

Therefore

$$x_{(3)}(t_1) = z_{0(3)}, \quad x_{(j)}(t_1) = 0, \quad j = 1, 2. \tag{5}$$

If the Evader uses an arbitrary control $v(\cdot)$, then by using (4) for his trajectory $y(t)$ we obtain

$$\begin{aligned} y_{(3)}(t_1) &= y_{0(3)} + \int_0^{t_1} v_{(3)}(t)\, dt \leqslant y_{0(3)} + \int_0^{t_1} \left|v_{(3)}(t)\right| dt \\ &\leqslant y_{0(3)} + t_1 = y_{0(3)} + |x_0 - z_0| < y_{0(3)} + z_{0(3)} - y_{0(3)} = z_{0(3)}. \end{aligned} \tag{6}$$

We have, from (5) and (6), $x_{(3)}(t_1) > y_{(3)}(t_1)$. We denote $a = x_{(3)}(t_1)$ and $b = y_{(3)}(t_1)$. Then $a > b$.

We continue constructing of the strategy for the Pursuer on the interval $[t_1, \infty)$ by the following way. Let r_0 be a root of the equation

$$a - b = R \ln \frac{R + \sqrt{R^2 - \tau^2}}{\tau}.$$

Since $a > b$, it is not difficult to check that the last equation has a unique solution $\tau = r_0 \in (0, R)$. Let

$$g(s) = \begin{cases} f(r_0) - (s - r_0)\sqrt{R^2 - r_0^2}/r_0, & 0 \leqslant s < r_0, \\ f(s), & r_0 \leqslant s \leqslant R, \end{cases}$$

where

$$f(s) = R\left(\ln \frac{R + \sqrt{R^2 - s^2}}{s} - \frac{\sqrt{R^2 - s^2}}{R}\right), \quad 0 < s \leqslant R.$$

Then

$$g(0) = a - b\,, \quad g(R) = 0\,, \tag{7}$$

$$\frac{dg(s)}{ds} = \begin{cases} -r_0^{-1}\sqrt{R^2 - r_0^2}, & 0 \leqslant s \leqslant r_0\,, \\ -s^{-1}\sqrt{R^2 - s^2}, & r_0 < s \leqslant R\,. \end{cases} \tag{8}$$

We replace the coordinates of the points x and y with the cylindrical coordinates $(\xi, \phi, z_{(3)})$. Then the third coordinates of these points are not changed. If $(r, \varphi, x_{(3)})$ and $(R, \psi, y_{(3)})$ are cylindrical coordinates of the points x and y, respectively, then

$$x_{(1)} = r\cos\varphi\,, \quad x_{(2)} = r\sin\varphi\,, \quad y_{(1)} = R\cos\psi\,, \quad y_{(2)} = R\sin\psi\,.$$

Therefore $r = \sqrt{x_{(1)}^2 + x_{(2)}^2}$, and from (1) we have

$$u_{(1)} = \dot r\cos\varphi - r\dot\varphi\sin\varphi\,, \qquad u_{(2)} = \dot r\sin\phi + r\dot\varphi\cos\varphi\,,$$
$$v_{(1)} = -R\dot\psi\sin\psi\,, \qquad v_{(2)} = R\dot\psi\cos\psi,$$

and the equality $x = y$ is equivalent to the system of equalities $r = R$, $\varphi = \psi$, $x_{(3)} = y_{(3)}$.

Denoting

$$\dot r = \tau\,, \quad \dot\varphi = \alpha\,, \quad \dot x_{(3)} = u_{(3)},$$
$$\dot R = 0\,, \quad \dot\psi = \beta\,, \quad \dot y_{(3)} = v_{(3)},$$

then vectors $(\tau, \alpha, u_{(3)})$ and $(0, \beta,\ v_{(3)})$ on the cylindrical coordinates systems are the parameters of control of the Pursuer and Evader respectively, and the conditions $|u| \leqslant 1$ and $|v| \leqslant 1$ take the forms

$$\tau^2 + r^2\alpha^2 + u_{(3)}^2 \leqslant 1\,, \tag{9}$$

$$R^2\beta^2 + v_{(3)}^2 \leqslant 1\,. \tag{10}$$

As $x_{(1)}(t_1) = x_{(2)}(t_1) = 0$, we consider that $\varphi(t_1) = \psi(t_1) = 0$.

Now on the cylindrical coordinate system we define a strategy U of the Pursuer at $t \geqslant t_1$

$$\tau = h\left(r, v_{(3)}\right), \quad \alpha = \beta, \quad u_{(3)} = v_{(3)} + h\left(r, v_{(3)}\right) g_r, \tag{11}$$

if $r \leqslant R$, and $\tau = 0$, $\alpha = 0$, $u_{(3)} = 0$ if $r > R$, where $g_r = dg/dr$ and

$$h(r, v_{(3)}) = \frac{-g_r v_{(3)} + \sqrt{g_r^2 v_{(3)}^2 + (1 + g_r^2)(1 - v_{(3)}^2) \frac{R^2 - r^2}{R^2}}}{1 + g_r^2}. \tag{12}$$

Here, we note that $|v_{(3)}| \leqslant 1$ and $r \leqslant R$ so the radicand is nonnegative, and also $h(r, v_{(3)})$ is a non-negative solution of the quadratic equation

$$h^2 \left(1 + g_r^2\right) + 2h g_r v_{(3)} - \left(1 - v_{(3)}^2\right) \frac{R^2 - r^2}{R^2} = 0. \tag{13}$$

In addition, as the function $g_r, 0 \leqslant r \leqslant R$, is continuous (see (8)), if the Evader uses an arbitrary control $(0, \beta(t), v_{(3)}(t))$, $t \geqslant t_1$, then the initial value problem

$$\begin{aligned}
&\dot{r} = h(r, v_{(3)}(t)), \quad \dot{\varphi} = \beta(t), \quad r(t_1) = \varphi(t_1) = 0, \\
&\dot{x}_{(3)} = v_{(3)}(t) + h(r, v_{(3)}(t)) g_r, \quad x_{(3)}(t_1) = a, \\
&\dot{R} = 0, \quad \dot{\psi} = \beta(t), \quad \dot{y}_{(3)}(t) = v_{(3)}(t), \quad \varphi(t_1) = 0, \quad y_{(3)}(t_1) = b,
\end{aligned} \tag{14}$$

has a unique absolutely continuous solution $(r(t), \varphi(t), x_{(3)}(t))$ and $(R, \psi(t), y_{(3)}(t))$, $t \geqslant t_1$.

First we show that for the strategy (11) the inequality (9) holds. If $r \leqslant R$, then, using the inequality $\beta^2 \leqslant R^{-2}(1 - v_{(3)}^2)$ (see (10)) and equality (13),

$$\begin{aligned}
&\tau^2 + r^2\alpha^2 + u_{(3)}^2 \\
&= h^2(r, v_{(3)}) + r^2\beta^2 + v_{(3)}^2 + 2v_{(3)} h(r, v_{(3)}) g_r + h^2(r, v_{(3)}) g_r^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant h^2(r, v_{(3)})(1+g_r^2) + \frac{r^2}{R^2}(1-v_{(3)}^2) + v_{(3)}^2 + 2v_{(3)}h(r, v_{(3)})g_r \\
&= h^2(r, v_{(3)})\left(1+g_r^2\right) + 2h(r, v_{(3)})g_r v_{(3)} - \left(1-v_{(3)}^2\right)\frac{R^2-r^2}{R^2} + 1 \\
&= 1\,.
\end{aligned}
$$

If $r > R$, then $\tau^2 + r^2\alpha^2 + u_{(3)}^2 = 0$.

We now show that, if the Evader uses an arbitrary control $(0, \beta(t), v_{(3)}(t))$, $t \geqslant t_1$, then for the solution of the initial problem (14) $r(t) = R$, $\alpha(t) = \beta(t)$ and $x_{(3)}(t) = y_{(3)}(t)$ at some $t \geqslant t_1$.

Let $t \geqslant t_1$ and $r < r_0$. Then using (8) we obtain from (13) that (for simplicity we do not write arguments t)

$$
\begin{aligned}
\dot{r}\frac{R}{r_0} - \frac{\sqrt{R^2-r_0^2}}{r_0}v_{(3)} &= (1+g_r^2)h(r, v_{(3)}) + g_r v_{(3)} \\
&= \sqrt{g_r^2 v_{(3)}^2 + (1+g_r^2)\left(1-v_{(3)}^2\right)\frac{R^2-r^2}{R^2}} \\
&= \sqrt{\frac{R^2-r_0^2}{r_0^2}v_{(3)}^2 + (1-v_{(3)}^2)\frac{R^2-r^2}{r_0^2}} \\
&= \sqrt{\frac{R^2-r_0^2}{r_0^2} + (1-v_{(3)}^2)\frac{r_0^2-r^2}{r_0^2}} \\
&\geqslant \frac{\sqrt{R^2-r_0^2}}{r_0}\,.
\end{aligned}
$$

Consequently,

$$
\dot{r} \geqslant \frac{\sqrt{R^2-r_0^2}}{R}v_{(3)} + \frac{\sqrt{R^2-r_0^2}}{R}\,.
$$

Integrating the last inequality from t_1 to t gives

$$
r(t) - r(t_1) \geqslant \frac{\sqrt{R^2-r_0^2}}{R}\left(y_{(3)}(t) - y_{(3)}(t_1) + t - t_1\right). \tag{15}
$$

As $y_{(3)}(t) \geqslant 0$, $t \geqslant 0$, and $r(t_1) = 0$, then (15) allows us to conclude that $r(t_2) = r_0$ at some t_2 such that

$$t_2 \leqslant \frac{Rr_0}{\sqrt{R^2 - r_0^2}} + y_{(3)}(t_1) + t_1 \,.$$

Let $t \geqslant t_2$. Note that if $r(t) < R$, then $\dot{r}(t) > 0$ and hence $r(t) \geqslant r(t_2)$. As the way shown above, combining (8) with (13) we obtain

$$\begin{aligned} \dot{r}\frac{R}{r} - \frac{\sqrt{R^2 - r^2}}{r}v_{(3)} &= (1 + g_r^2)h(r, v_{(3)}) + g_r v_{(3)} \\ &= \sqrt{\frac{R^2 - r^2}{r^2}v_{(3)}^2 + (1 - v_{(3)}^2)\frac{R^2 - r^2}{r^2}} \\ &= \frac{\sqrt{R^2 - r^2}}{r} \,. \end{aligned}$$

Consequently,

$$\dot{r} = \frac{\sqrt{R^2 - r^2}}{R}v_{(3)} + \frac{\sqrt{R^2 - r^2}}{R} \,.$$

Therefore

$$\frac{\dot{r}}{\sqrt{R^2 - r^2}} = \frac{1}{R}\left(v_{(3)} + 1\right) .$$

We integrate these equality to have

$$\arcsin\frac{r(t)}{R} - \arcsin\frac{r_0}{R} = \frac{1}{R}\left(y_{(3)}(t) - y_{(3)}(t_2) + t - t_2\right) . \qquad (16)$$

As $y_{(3)}(t) \geqslant 0$ and $\arcsin \frac{r_0}{R} > 0$, by (16) there exists a number $t_3 < \frac{\pi R}{2} + t_2 + y_{(3)}(t_2)$ such that $\arcsin \frac{r(t_3)}{R} = \frac{\pi}{2}$ and so $r(t_3) = R$.

Now consider the third coordinate of point x. From (11) we obtain

$$x_{(3)}(t_3) = x_{(3)}(t_1) + \int_{t_1}^{t_3} u_{(3)}(t)\, dt$$

$$= x_{(3)}(t_1) + \int_{t_1}^{t_3} \big(v_{(3)}(t) + h(r(t), v_{(3)}(t)) g_r(r(t))\big)\, dt$$

$$= x_{(3)}(t_1) + y_{(3)}(t_3) - y_{(3)}(t_1) + \int_{t_1}^{t_3} \dot{r}(t) g_r(r(t))\, dt$$

$$= a + y_{(3)}(t_3) - b + g(r(t_3)) - g(r(t_1)).$$

Taking into account the equalities $r(t_1) = 0$ and $r(t_3) = R$, and (7), from the last equality

$$x_{(3)}(t_3) = a + y_{(3)}(t_3) - b + g(R) - g(0) = y_{(3)}(t_3). \tag{17}$$

From the equalities $\varphi(t_1) = \psi(t_1) = 0$ and $\alpha(t) = \beta(t)$, $t \geqslant t_1$, we have $\varphi(t) = \psi(t) = 0$ for all $t \geqslant t_1$. Then this equality together with (17) implies $x(t_3) = y(t_3)$.

In the final part of the proof of the theorem, we observe that using estimates for t_2 and t_3 above and inequality $y_{(3)}(t) \leqslant y_{0(3)} + t$, $t \geqslant 0$, we obtain the following estimate for the time

$$T(x_0, y_0) = t_3 \leqslant \frac{R\pi}{2} + \frac{2Rr_0}{\sqrt{R^2 - r_0^2}} + 4|x_0 - z_0| + 3y_{0(3)}$$

for which the pursuit can be completed.

The proof of Theorem 7 is complete. ♠

Remark 8. If $y(t) \in \{z \in M \mid 0 \leqslant z_{(3)} \leqslant a\}$, $t \geqslant 0$, then pursuit can be completed in the game (1)–(3) from any initial positions.

The General case: $N = \mathbb{R}^3$ Here we study the case (1)–(3), when the Pursuers move throughout the space and the Evader moves on the M. We obtain the following corollary of Theorem 7.

Corollary 9. *If $m = 1$, then for any initial position (x_0, y_0), $x_0 \neq y_0$, pursuit cannot be completed in the differential game (1)–(3).*

Theorem 10. *1. If there exist some indices $i, j \in \{1, 2, \dots, m\}$ such that $x_{i0(3)} < y_{0(3)} < x_{j0(3)}$, then pursuit can be completed for some time $T(x_{10}, x_{20}, \dots, x_{m0}, y_0)$ in the game (1)–(3).*

2. If $i, j \in \{1, 2, \dots, m\}$ do not exist to satisfy $x_{i0(3)} < y_{0(3)} < x_{j0(3)}$, then evasion is possible.

Proof:

1. Let there exist indices $i, j \in \{1, 2, \dots, m\}$ to satisfy $x_{i0(3)} < y_{0(3)} < x_{j0(3)}$. Without loss of generality we assume that $i = 1$, $j = 2$ and $y_{0(3)} = 0$. We introduce the fictitious Evaders $\bar{y}_1$ and $\bar{y}_2$

$$\bar{y}_{i(j)}(t) = y_{i(j)}(t), \quad j = 1, 2, \quad \bar{y}_{i(3)}(t) = (-1)^i \left|y_{i(3)}(t)\right|, \quad t \geqslant 0.$$

The points $\bar{y}_1$ and $\bar{y}_2$ move with maximal speed equal to 1 on the half cylinders $M_- = \left\{z \in M \mid z_{(3)} \leqslant 0\right\}$ and $M_+ = \left\{z \in M \mid z_{(3)} \geqslant 0\right\}$ respectively.

Moreover one of the points $\bar{y}_1(t)$ and $\bar{y}_2(t)$ coincides with $y(t)$, $t \geqslant 0$. So if $\bar{y}_i(t) = x_i(t)$ for both $i = 1$ and $i = 2$ at some $t > 0$, then $y(t) = x_i(t)$ at some $i = 1, 2$. By Theorem 7 there exist strategies U_1 and U_2 of pursuers P_1 and P_2, and number t_* such that $\bar{y}_i(t) = x_i(t)$, $i = 1, 2$, $t \geqslant t_*$. Therefore, pursuit can be completed at some time $T(x_0, y_0) \leqslant t_*$.

2. In this case we assume that $x_{i0(3)} \leqslant y_{0(3)}$, $i = 1, 2, \dots, m$. We show similarly to the beginning part of the proof of Theorem 7 that the Evader using the control $v(t) = (0, 0, 1)$, $t \geqslant 0$ ensures $x_i(t) \neq y(t)$, $t \geqslant 0$, $i = 1, 2, \dots, m$.

The proof of Theorem 10 is complete. ♠

The case when all players move on the surface of the cylinder We consider the differential game (1)–(3) when $N = M$. This means all Pursuers as well as the Evader move on the surface of the cylinder M.

Theorem 11. *1. If there exist indices $i, j \in \{1, 2, \ldots, m\}$ with the property that $x_{i0(3)} < y_{0(3)} < x_{j0(3)}$ and $m \geqslant 3$, then pursuit can be completed in the game (1)–(3) for some time $T(x_{10}, x_{20}, \ldots, x_{m0}, y_0)$.*

2. If either $x_{i0(3)} \leqslant y_{0(3)}$ or $x_{i0(3)} \geqslant y_{0(3)}$ for all $i \in \{1, 2, \ldots, m\}$, then evasion is possible in the game (1)–(3).

Proof: To prove Theorem 11 we reduce the game (1)–(3) on the cylinder M to the specific game in the plane $\mathbb{R}^2$. Such a reduction is conducted by the multivalue mapping that is inverse to the universal covering $F : \mathbb{R}^2 \to M$ which is local isometry (Nikulin and Shafarevich [15]). If $z \in M$, then the aggregate of its preimages $F^{-1}(z)$ consists of the class of denumerable number of points equivalent to each other $\ldots z^{-2}, z^{-1}, z^0, z^1, z^2, \ldots \in \mathbb{R}^2$ such that

$$z^j_{(1)} = z^0_{(1)} + 2j\pi R\,, \quad z^j_{(2)} = z^0_{(2)}, \quad j = \pm 1, \pm 2, \ldots\,. \tag{18}$$

Let $F^{-1}(y) = \{y^j \mid j \in Z\}$, $F^{-1}(x_i) = \{x^j_i \mid j \in Z\}$, $i = 1, 2, \ldots, m$. Then, the equality $x_i = y$ is equivalent to $x^j_i = y^k$ for some $j, k \in Z$. This property allows us to reduce the formulated game (1)–(3) to the game in the Euclidean plane with many groups of Pursuers and one Evader. Every group consists of countably many pursuers controlled by one parameter. The equations of motions are

$$\dot{x}^j_i = u_i\,, \quad x^j_i(t_0) = x^j_{i0}\,, \quad i = 1, 2, \ldots, m\,, \quad j \in Z\,, \quad \dot{y} = v\,, \quad y(t_0) = y_0, \tag{19}$$

where $x^j_i, y, u_i, v \in \mathbb{R}^2$; u_i and v are control parameters of the group of Pursuers $F(x_i)$ and the Evader y, respectively, and they satisfy the conditions

$$|u_i| \leqslant 1\,, \quad i = 1, 2, \ldots, m\,, \quad |v| \leqslant 1\,. \tag{20}$$

Pursuit can be completed in game (18)–(20) if $x_i^j(t) = y(t)$ for some $i \in \{1, 2, \ldots, m\}$, $j \in Z$ and $t > 0$. ♠

According to the properties of F mentioned above, pursuit is completed in both games (1)–(3) and (18)–(20) simultaneously. Therefore, we study the game (18)–(20).

Theorem 12. *1. If there exist indices $i, j \in \{1, 2, \ldots, m\}$ with the property that $x^0_{i0(2)} < y_{0(2)} < x^0_{j0(2)}$ and $m \geqslant 3$, then pursuit can be completed in the game (18)–(20) for some time $T(x_{10}, x_{20}, \ldots, x_{m0}, y_0)$.*

2. If either $x^0_{i0(2)} \leqslant y_{0(2)}$ or $x^0_{i0(2)} \geqslant y_{0(2)}$ for all $i \in \{1, 2, \ldots, m\}$, or $m < 3$, then evasion is possible in the game (18)–(20).

Proof:

1. Let $x^0_{i0(2)} < y_{0(2)} < x^0_{k0(2)}$ for some $i, k \in \{1, 2, \ldots, m\}$ and $m \geqslant 3$. We prove that pursuit can be completed in the game (18)–(20). Without loss of generality we assume $x^0_{10(2)} < y_{0(2)} < x^0_{20(2)}, x^0_{i0(1)} < y_{0(1)}$, $i = 1, 2$, and $m = 3$. Then, by (18) there exists $j \in Z$ such that $y_0 \in \operatorname{intco}\{x^0_{10}, x^0_{20}, x^j_{30}\}$, where $\operatorname{intco} A$ denotes the interior convex hull of the set A. We consider only motions of the pursuers x^0_1, x^0_2 and x^j_3, therefore, for short we write without superscripts, as x_1, x_2 and x_3. Now the motions of the Pursuers are described by the equations $\dot{x}_i = u_i$, $|u_i| \leqslant 1$, $i = 1, 2, 3$.

   Now construct a strategy of the Pursuers as strategy of parallel approach (p-strategy)
$$U_i(v) = v + \lambda_i(v) e_i\,, \tag{21}$$
   where
$$e_i = \frac{y_0 - x_i}{|y_0 - x_i|}\,, \quad \lambda_i(v) = -\langle v, e_i\rangle + \sqrt{1 - |v|^2 + (\langle v, e_i\rangle)^2}\,, \quad i = 1, 2, 3\,.$$

and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^2$.

As $|v| \leqslant 1$ and $\lambda_i(v) \geqslant 0$, $i = 1, 2, 3$, from the condition that $y_0 \in \operatorname{int} \operatorname{co}\{x_{10}, x_{20}, x_{30}\}$,

$$\min_{|v|\leqslant 1} \sum_{i=1}^{3} \lambda_i(v) > 0 .$$

We denote

$$T = \left(\sum_{i=1}^{3} |x_{i0} - y_0| \right) \left(\min_{|v|\leqslant 1} \sum_{i=1}^{3} \lambda_i(v) \right)^{-1} .$$

Let the Evader use an arbitrary control $v(\cdot)$ and the Pursuers uses the p-strategy and $y(\cdot)$, $x_i(\cdot)$, $i = 1, 2, 3$, his corresponding trajectories. Observe that if the Pursuers use the p-strategy, then the vectors $x_i(t) - y(t)$ and e_i are parallel at all time (Petrosyan [3, 4]).

We assume the contrary, that is $x_i(t) \neq y(t)$, $t \in [0, T]$. Then, for $i = 1, 2, 3$ by (21) we have

$$\begin{aligned} \frac{d}{dt} |x_i(t) - y(t)| &= \left\langle \frac{x_i(t) - y(t)}{|x_i(t) - y(t)|}, u_i(v(t)) - v(t) \right\rangle \\ &= \left\langle \frac{x_i(t) - y(t)}{|x_i(t) - y(t)|}, v(t) + \lambda_i\,(v(t))\, e_i - v(t) \right\rangle \\ &= -\lambda_i(v(t)). \end{aligned}$$

Hence,

$$\sum_{i=1}^{3} \frac{d}{dt} |x_i(t) - y(t)| = -\sum_{i=1}^{3} \lambda_i(v(t)) \leqslant -\min_{|v|\leqslant 1} \sum_{i=1}^{3} \lambda_i(v).$$

Integrating the last inequality from 0 to T, we obtain

$$0 < \sum_{i=1}^{3} |x_i(t) - y(t)| \leqslant \sum_{i=1}^{3} |x_{i0} - y_0| - T \min_{|v|\leqslant 1} \sum_{i=1}^{3} \lambda_i(v) = 0 ,$$

which contradicts our assumption. Thus pursuit can be completed for the time T.

2. Let $x^0_{i0(2)} < y_{0(2)}$ for all $i = 1, 2, \ldots, m$. Then as in the proof of Theorem 7, we prove that the evasion is possible if the Evader uses the control $v(t) = (0, 1), t \geqslant 0$.

Now we consider the case $m < 3$. Let $m = 2$ and $x^0_{10(2)} < y_{0(2)} < x^0_{20(2)}$ (otherwise the above arguments imply that evasion is possible). We prove that for any initial position, evasion is possible.

We construct the strategy of the Evader by conditions

$$\langle V, x_1^{j_1} - x_2^{j_2} \rangle = 0\,, \quad \langle V, x_i^{j_i} - y \rangle \leqslant 0\,, \quad i = 1, 2\,, \tag{22}$$

where $x_i^{j_i} \in \{x_i^j \mid j \in Z\} = F^{-1}(x_i)$ and

$$\min_{j \in Z} \left| y - x_i^j \right| = \left| y - x_i^{j_i} \right|, \quad i = 1, 2\,. \tag{23}$$

By (18) and (23),

$$\left| y - x_i^j \right| \geqslant \pi R\,, \quad j \neq j_i\,, \tag{24}$$

where $j_i, i = 1, 2$, satisfy (23).

Let $\Delta = \{0 = t_0, t_1, t_2, \ldots\}$ be a partition of the interval $[0, \infty)$ with $t_i = i\pi R/4$. We consider the trajectory $y(\cdot)$ generated by the strategy V and the partition Δ. We assume that $y(t) \neq x_i^j(t)$ for all $i = 1, 2, j \in Z$, and $t \in [0, t_k]$, where k is some non-negative integer and then prove that $y(t) \neq x_i^j(t)$ for all $i = 1, 2, j \in Z$, and $t \in [t_k, t_{k+1}]$. Without loss of generality we consider that $x_i^{j_i} = x_i^0$, that is, $j_i = 0, i = 1, 2$. Then by (22) we have

$$\begin{aligned} \left| y(t) - x_i^0(t_k) \right| &= |y(t_k) - x(t_k) + (t - t_k)V| > |(t - t_k)V| \\ &= t - t_k \geqslant \left| \int_{t_k}^{t} u_i(s)\, ds \right| \end{aligned}$$

$$= \left|x_i^0(t) - x_i^0(t_k)\right|, \quad t_k \leqslant t \leqslant t_{k+1}, \quad i = 1, 2.$$

Consequently, $y(t) \neq x_i^0(t)$, $i = 1, 2$, $t \in [t_k, t_{k+1}]$. For $j \neq 0$ by using (24) and $t_i = i\pi R/4$ we obtain

$$\begin{aligned} \left|y(t) - x_i^j(t)\right| &= \left|y(t_k) - x_i^j(t_k) + (t - t_k)V - \int_{t_k}^t u_i(s)\, ds\right| \\ &\geqslant \left|y(t_k) - x_i^j(t_k)\right| - |(t - t_k)V| - \left|\int_{t_k}^t u_i(s)\, ds\right| \\ &\geqslant \pi R - (t - t_k) - (t - t_k) \\ &\geqslant \frac{\pi R}{2}, \quad t \in [t_k, t_{k+1}], \quad i = 1, 2. \end{aligned}$$

This implies that $y(t) \neq x_i^j(t)$, $t \in [t_k, t_{k+1}]$, $j \in Z$, $i = 1, 2$. Therefore, by induction the inequalities $y(t) \neq x_i^j(t)$ hold for any $t \in [0, \infty)$, $j \in Z$, and $i = 1, 2$.

The proof of Theorem 12 is complete. ♠

Acknowledgements: We thank Prof. Dr. Abdulla Azamov, Institute of Mathematics and Information Technology of Academy Science of Republic Uzbekistan and Assoc. Prof. Dr. Gafurjan Ibragimov, Department of Mathematics, Universiti Putra Malaysia, for helpful discussions.

References

[1] Isaacs, R. *Differential Games. A Mathematical Theory with Applications to Warfare and Pursuit, Control, and Optimization*. New York: Wiley, 1963. E2

[2] Petrov, N. N. A problem of group pursuit with phase constraints. *J. Appl. Math. Mech.* 1988, **52**, No 6, 1030–1033. E2

[3] Petrosyan, L. A. Survival differential game with many participants. *Dokl. Akad. Nauk USSR.* 1965, **161**, No 2, 285–287. E2, E16

[4] Petrosyan, L. S. *Differentsial'nye igry presledovaniya* (Pursuit Differential Games). Leningrad (SPb): Leningr.State Univ(SPbSU), 1977. E2, E16

[5] Pshennichnyi, B. N. Simple pursuit of several targets. *Kibernetika.* 1976, No 3, 145–146. E2

[6] Chernous'ko, F. L. A problem of evasion of several pursuers. *J. Appl. Maths Mechs.*1976, **40**, No 1, 14–24. E2

[7] Ivanov R. P. Simple pursuit-evasion on a compact. *Dokl. Akad. Nauk SSSR.* 1980, **254**, No 6, 1318–1321. E2

[8] Melikyan, A. A. and Ovakimyan, N. V. Singular trajectories in the problem of simple pursuit on a manifold. *J. Appl. Math. Mech.* 1991, **55**, No 1, 42–48. E2

[9] Melikyan, A. A. and Ovakimyan, N. V. *Differential Games of Simple Pursuit and Approach on Manifolds.* Institute of Mechanics, National Academy of Sciences of Armenia. Yerevan. Preprint, 1993. E2

[10] Kuchkarov, A. Sh. The problem of optimal approach in locally euclidean spaces. *Automation and Remote Control.* 2007, **68**, No 6, 974-978. E2

[11] Kuchkarov, A. Sh. A simple pursuit–evasion problem on a ball of a Riemannian manifold. *Mathematical Notes.* 2009, **85**, No 2, 190–197. E2

[12] Azamov, A. On a problem of escape along a prescribed curve. *J. Appl. Math. Mech.* 1982, **46**, No 4, 553–555. E2

[13] Kuchkarov, A. Sh. and Rikhsiev, B. B. on the solution of a pursuit problem with phase constraints. *Automation and Remote Control.* 2001, **62**, No 8, 1259–1262. E2

[14] Ibragimov G. I. A game problem on a closed convex set. *Siberian advances in mathematics.*, 2002, **12**, No 3, 16–31. E2

[15] Nikulin, V. V. and Shafarevich, I. R. *Geometriya i Gruppy* (Geometric and Groups). Moscow: Nauka, 1983. E14

Author addresses

1. **A. Sh. Kuchkarov**, Institute of Mathematics and Information Technology of Academy Science of Republic Uzbekistan and INSPEM, Universiti Putra Malaysia, Serdang, Malaysia.
2. **M. H. Risman**, Dept. Maths, Universiti Putra Malaysia, Serdang, Malaysia.
   mailto:risman@math.upm.edu.my
3. **A. H. Malik**, INSPEM & Dept. Maths, Universiti Putra Malaysia, Serdang, Malaysia.