Tractable approximations to multistage decisions in air defence scenarios

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Abstract

Simulations are commonly used to investigate the control and resource allocation problems associated with pitting aircraft against ground based air defences. Such simulations rapidly become computationally intractable as units are added. Previous work described an envelope method that retains computational tractability if the lowest and highest cost target sequences can be defined a priori and used to establish solution bounds. This approach must be modified to be applied to the more common case where there are no obvious best or worst sequences of targets. We show that these bounding sequences can be approximated by using binary comparisons and by basing decisions on a heuristic. This approach compares well with exact results in some computationally tractable situations.

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1 Introduction

Suppression of enemy air defence is a multi-stage resource allocation and control problem requiring optimisation of an N-stage attack process. This problem has been approached using competitive games, with the dual intentions of developing a method for real situations and of providing summary information for the higher level problem of capability planning. Relevant approaches include decomposing the spatio-temporal problem into a temporal problem and a spatial problem, with simulation used to determine a Nash solution to the spatial problem and war gaming to solve the temporal problem [2]; using a Nash approach to dynamically reassign resources as conditions change [10]; taking a recursive approach to a three layer problem and optimising for two layers at each step [4]; using one step Nash optimisation [3]; and using games and simulations, coupled with approximations to bring some reduction in dimensionality [5]. These work for small values of N, but as the number of states increases they become computationally intractable [5]. In all of these models, optimisation refers to minimisation of maximum losses.

We previously showed that a direct link between the tactical level and the capability planning level allows the attack state to be decoupled from the defence state, reducing the game to a single Markov model, and enables the use of an envelope method, for summary information, that brings a dramatic reduction in the number of states that need to be considered [6, 7, 8]. However the envelopes are based on prior knowledge of the sequences of targets that cause the N-stage costs to be close to minimum and maximum. In this approach, optimisation refers to minimisation of expected losses, but the essential characteristics of the optimisation problem are unchanged. This approach has been shown to work by applying it to a strategic problem using historical data [8]. In that case the low and high cost sequences were obvious.

The problem addressed here is the generation of computationally feasible approximate methods of solution to the multistage cost minimisation of air defence suppression. We look at a case where there is no spatial variation in the threat to attackers within an engagement zone but allow spatial variation in the locations of defence units and mutual support between all defence units. Complete optimisation can be achieved by exhaustive elaboration or via dynamic programming, but this becomes intractable in problems with more than (approximately) ten stages, so that some real problems are tractable while others are not.

We explore the problem via binary comparisons, effectively modelling the N-stage problem by a series of 2-stage problems, and demonstrate that the stage decisions are coupled, in the sense that cost minimising sequences include high cost choices at individual stages, and context dependent, where the context is supplied by air defence sites other than the two being considered. We also look at the effects of variation of problem parameters. Despite context dependence, binary comparisons can still be used to determine preference orders among options as long as the scoring process used takes both background context and local context into account [9]. We follow this background plus foreground approach and develop a simple heuristic that performs well. To date, we have tested it against exhaustive elaboration for up to five stages.

2 Problem definition and exploration

We consider worst case scenarios for attackers of air defence complexes where all defence sites have entirely overlapping engagement zones, henceforth called the engagement zone. Competent attackers will always use topography and electronic warfare to create low risk corridors to their targets, so we assume they have done their best to minimise the extent of the engagement zone. While attackers will attempt to isolate air defence assets and turn an overlapping engagement zone into a set of engagement zones to enable piecemeal engagement of these non-mutually supporting sites, we assume modern networking of sensors and launchers across sites negates such attempts. Furthermore, sensor rather than missile range is typically the limiting factor in engagement, so effective networking of sensors makes the overlapping of missile engagement zones a realistic assumption.

Attackers target one site at each stage of their attack. They have the dual objectives of using an N-stage process to disable all air defence sites and of minimising their own total losses. Standard aircraft survivability equations are transformed to show attacker survivability over the N-stage process depends on the location of the targeted site and the lethalities of all active sites. Expected losses, or costs, to the attackers are derived for a single stage. For two sites there are two possible orders of attack and costs are calculated for each. We demonstrate that the cost minimising order of attack varies with the locations and lethalities of the two sites. For the N site case the other $N-2$ sites provide a context for the order of attack decision for any pairing and this context strongly affects the decision. Finally, we show that a threshold for acceptable losses will impose constraints on the possible values of lethalities and locations.

2.1 Survivability equations

The probability of an aircraft surviving a sortie is [1]

$$p_{s,s} = \prod_i (1 - p_{K,e})_i^{e_i} \,, \tag{1}$$

where $1 - p_{K,e}$ is the probability the aircraft survives an encounter with the ith weapon type and e_i is the number of such encounters during the sortie. When the sortie is to attack a target k and return to base, the number of encounters will depend on the distance to the target and the opportunities the trip will create for encounters with a weapon. In general the number of encounters n_{ijk} will depend on the interaction between the aircraft j and the weapon i as well as on the exposure that is associated with a trip to the target k. The probability of kill per encounter $p_{K,e}$ in each term specifically refers to an encounter between an aircraft j and a weapon i and we substitute p_{ij} to explicitly reflect those connections. The survival probability is for a trip by aircraft j to target location k and we also represent this explicitly as $p_{s,jk}$. This gives an expression,

$$p_{s,jk} = \prod_i (1 - p_{ij})_i^{n_{ijk}} \,, \tag{2}$$

that is difficult to work with because of the unnecessary complexity associated with n_{ijk}. Regardless of the terrain or of the capability set supporting the aircraft, we can represent n_{ijk} as the product of two factors

$$n_{ijk} = (s_i f_{ij}) \left(\frac{d_{jk}}{v_j} \right) \,, \tag{3}$$

where the first factor is independent of k and the second is independent of i. The terms are the firing rate s_i for weapon i, the fraction f_{ij} of shots from i that are directed at an aircraft of type j, the distance d_{jk} that must be traversed in danger on the way to target k, and the velocity v_j of an aircraft of type j. We then define the lethality l_{ij}

$$(1 - l_{ij}) = (1 - p_{ij})^{s_i f_{ij}} \,, \tag{4}$$

where l_{ij} is the probability, for an aircraft of type j, of being disabled by defence site i in unit time. The probability of survival

$$p_{s,jk} = \left[\prod_i (1 - l_{ij})\right]^{d_{jk}/v_j} . \tag{5}$$

Thus the survival probability depends on the lethalities of all active sites and the time spent in danger in reaching the location of the target site.

2.2 Cost equation

The expected losses for a single unsuccessful attack and return to base are

$$u_{jk} = m_j (1 - p_{s,jk}^2) \,, \tag{6}$$

based on the probability that an aircraft will not survive the combined trip from base to air defence site and return, when m_j is the number of aircraft in the attack team. The expected number of attacks until a target is disabled is

$$r_{jk} = \frac{1}{p_{s,jk} p_{h,jk}} \,, \tag{7}$$

where $p_{h,jk}$ is the conditional probability that the team will disable the target k in a single attack, given that they are in a position to launch their weapons. The expected cost can then be approximated by $c_{jk} = u_{jk} r_{jk}$, which is an overestimate because the trip to base after a successful attack will have an increased probability of survival. The overestimation can be eliminated by a correction term which we do not include here but intend to include in further developments of this approach. The cost can be represented as a function of the probability p_k of an aircraft surviving for unit time while enroute to target k, where we have removed the explicit reference to aircraft type j as only one aircraft type is involved. The survival probability for unit time p_k is

$$p = \prod_i (1 - l_i)\,, \tag{8}$$

and the expected cost of disabling target k is

$$c_k(p) = \frac{m}{p_{h,k}} \left(p^{-t_k} - p^{t_k} \right)\,, \tag{9}$$

where $t_k = d_k/v$ is the time spent in danger on the journey to site k.

2.3 Binary cost equation

We represent an N-stage process by a set of 2-stage, or binary, processes where just two defence sites are considered explicitly, with the contribution of the other sites to the survivability per unit time being represented by a background survivability. For simplicity, the sites being considered are referred to as sites 1 and 2. The survival probability per unit time at the beginning is a modified version of equation (8):

$$p = (1 - l_1)(1 - l_2) p_{sr}\,, \tag{10}$$

where p_{sr} is the background probability of survival, due to the effect of all those sites that are not being explicitly considered:

$$p_{sr} = \prod_{i=3}^{N} (1 - l_i)\,. \tag{11}$$

If site 1 was attacked until disabled, attacks on site 2 would have a probability of survival for unit time of $p/(1-l_1)$, so that the cost of disabling site 1 and then disabling site 2 is

$$C_{12}(p) = c_1(p) + c_2(p/(1-l_1))\,, \tag{12}$$

where $c_k(p)$ is given by equation (9). The alternative cost, for disabling site 2 first, followed by site 1 is

$$C_{21}(p) = c_2(p) + c_1(p/(1-l_2))\,. \tag{13}$$

2.4 Inter-stage coupling

To illustrate the coupling between decisions at each stage, we consider the differences between the two binary cost equations (12) and (13) in two cases, with the first having lethality $l_1 = 0.05$ and the second having lethality $l_2 = 0.2$ in each case. The two cases vary in the time that must be spent in danger in order to attack defence site 1. This time is given by d_1/v and, in case 1, t_1 is one time unit, compared to four time units in case 2. The time in danger for attacks on site 2, t_2, is five time units in each case. These times are exponents in the cost equation (9) that is used to define the two binary cost equations (12) and (13). This situation is represented in Table 1, with the total cost of attacking the two defence sites in different orders $(1 \to 2, 2 \to 1)$ shown in each case. For case 1, the expected cost of attacking site 1 first is $c_1(p) = 1.11$, while attacking site 2 first would cost $c_2(p) = 7.38$. Once the initial site is disabled, the cost of attacking the remaining site is 0.21 for site 1 and 5.45 for the site 2. Thus, for case 1 the cost of a two-stage attack is minimised by targeting the low-cost site first $(1 \to 2)$, that is, $C_{12} < C_{21}$. In case 2, site 1 has moved deeper into the engagement zone and the two-stage expected cost is minimised by maximising the expected cost at stage one $(2 \to 1)$.

Table 1: Inter-stage coupling of optimal decisions.

		Case 1	Case 2
Distances	t_1	1	4
	t_2	5	5
Costs	$1 \to 2$	6.56	10.78
	$2 \to 1$	7.59	8.21

2.5 Dependence on background context

Decisions are often influenced by context and this is particularly true here with the decisions being binary and the context being supplied by the background defence sites (those not being explicitly considered). However, decisions may be independent of context and this also occurs here. We define the decisions that depend on context and those that are context free, and for those that depend on context we define a decision surface, where the decision changes as the surface is crossed. At the decision surface, the costs of the two options are equal. We consider a two dimensional subset of variables with the decision surface becoming the decision boundary, which is implicitly defined by the equation for the surface, and we demonstrate the effect of context by graphing the different decision boundaries that result from different context survivabilities.

When all other things are equal the binary choice will depend on lethality and time. If the less lethal site is closer than the more lethal site, that is, if $l_1 < l_2$ and $t_1 < t_2$, then we have a decision problem, whereas if the more lethal site is either closer or an equal time away, then it should be attacked first. If the lethalities are equal, the closer site (smaller t) should be attacked first.

For a binary decision problem we choose the order that gives the lesser cost, where the costs are calculated from C_{12} and C_{21}, equations (12) and (13).

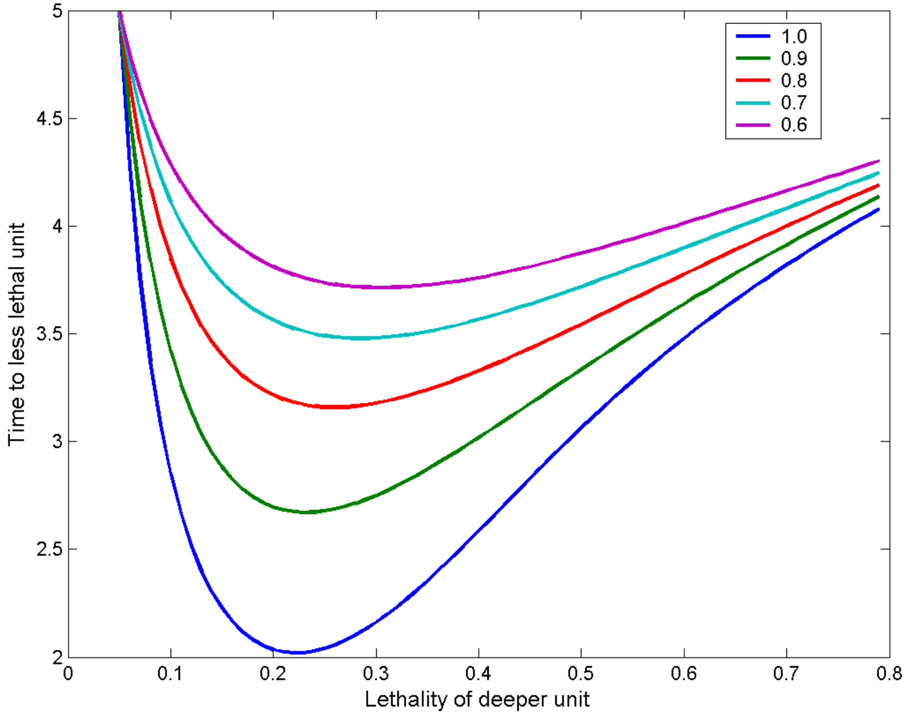

FIGURE 1: Effect of background context on locus of binary decision boundary, for context survivabilities (p_{sr}) of 1.0, 0.9, 0.8, 0.7 and 0.6, for the case where $l_1 = 0.05$ and $t_2 = 5$.

We define a decision surface, where the decision changes as the surface is crossed, by

$$g(p_{sr}, l_1, l_2, t_1, t_2) = C_{12} - C_{21} = 0\,. \tag{14}$$

Without loss of generality we take $l_2 \geq l_1$ and $t_2 \geq t_1$. Consider the two-dimensional decision boundary defined by $g = 0$ in the (l_2, t_1) plane. For constant values of p_{sr}, l_1, and t_2, $g = 0$ defines an implicit relationship between l_2 and t_1, allowing the decision boundary to be defined numerically, for example via iterative use of the Newton method. A set of values for p_{sr} defines a set of decision boundaries that illustrate the effect of context.

In Figure 1 values of (l_2, t_1) above the graph indicate that site 2 should be attacked first, with site 1 being preferred below the graph. The bottom line is for $p_{sr} = 1$, while the lines above it are for $p_{sr} = 0.9, 0.8, 0.7, 0.6$. Note the considerable area where a decision would go different ways in different background contexts. The graph is shown for $t_2 = 5$, $l_1 = 0.05$, raising the question of what happens if those values are different. The full results of our trials are too complex to be shown in this paper but the shapes and sizes of the curves are relatively insensitive to the value of t_2.

2.6 Cost threshold

Although the cost equations (12) and (13) and the equation for the decision boundary (14) can be solved for all $0 < (p_{sr}, l) < 1$ and $t > 0$, there will be a practical threshold, determined by the maximum acceptable loss c_m in disabling a single defence site. Using the form $p^t = e^{t\ln(p)}$, equation (9) transforms to

$$c_k(p) = \frac{2m}{p_{h,k}} \sinh\{t[-\ln(p)]\}\,. \tag{15}$$

Then $c_k(p) \leq c_m$ for each k, gives

$$-\sinh^{-1}\left(\frac{c_m p_h}{2m}\right) \leq t \ln p \leq 0\,. \tag{16}$$

Capability sets would be rejected if they produced higher levels of cost. The number of stages would be one of the factors determining the value of c_m.

Further development of this heuristic approach may involve approximations and these are likely to require convergence conditions. Regions of convergence would need to be compared with regions defined by the acceptable level of losses.

3 Approximate Solutions

A strong coupling between decision and context can be dealt with by using a weighting process to combine the effect of background context with the effect of local context [9]. We follow this approach by balancing the cost of disabling a unit in stage 1 with the benefit that disablement confers on the costs of disabling each other unit at stage 2. If defence site i is attacked first, the cost for stage 1 will be $c_i(p)$ and the sum of the cost reductions in attacking all of the other units at stage 2 is

$$S_i = \sum_{j \neq i} \left(c_j(p) - c_j(p/(1 - l_i)) \right) . \tag{17}$$

The score, B_i, for defence site i is $B_i = S_i - c_i(p)$. Defence units are chosen in descending order of scores. Therefore, this is a ranking method. The comparisons between the optimal sequences and the predictions of the heuristic are intended to explore the cases where the heuristic works well and those where it does not. In each set of comparisons we start with a case where the approximate and numerical solutions both agree that the attack order should be $1, 2, \ldots, N$ and then progressively increase the distance of the $N - 1$ site until both methods agree on swapping the order of $N - 1$ and N in the attack sequence, noting when each method changes the order. This process is repeated for sites $N - 2$ down to 1. Results show that the approximate solution agrees with the numerically derived ordering except that it lags behind the numerical optimum in moving more lethal sites to an earlier stage. It requires a higher expected value for the time in danger, with the differences varying between 0.2 and 0.6 in the value of t needed for the change. When $N = 5$ the worst error case gave a minimum cost just under 2% higher than that found through enumeration. This difference represented 3% of the difference between minimum and maximum costs. The modal value for the error in the minimum cost was 0.1%. The maximum error, when represented as a percentage of the difference between the minimum and maximum costs, was under 10%, but this was achieved for very small values of t where the minimum and maximum costs were very close to each other.

4 Conclusion

The use of binary comparisons with a background context represents a computationally feasible approximate solution to the N-stage resource allocation problem typically used to model suppression of enemy air defences. This yields a complexity benefit by reducing an $O(N!)$ problem to one that is $O(N^2)$ while introducing errors in the costs that are, in the worst case, less than 2% for a five stage problem. This error is small compared to the human errors in estimating $p_{K,e}$, which are difficult to quantify. Further work could use a Taylor series expansion of equation (17) to achieve a first order expression for the benefit, with convergence to be tested.

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