

A practical algorithm for the computation of tree felling times

D. S. Holloway¹

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Abstract

The time taken for a tree to fall when felled can be expressed in closed form in terms of elliptic integrals. However, this form of the solution is of limited practical use to the non-mathematician. A straightforward numerical method is presented that can be programmed into a spreadsheet in a few lines without requiring any high level programming or mathematical skills. Good accuracy for practical purposes is obtained without any iteration, but accuracy is further improved with two or three iterations, each involving only one or two extra cells in the spreadsheet. Results are compared with the analytic solution. This article demonstrates how a challenging real problem is accurately solved numerically with basic office tools and limited mathematics.

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1 Introduction

The subject of this article is depicted by the free body diagram in Figure 1. When a tree is felled its motion to the ground occurs in two stages: the initial rotation about its point of contact with the stump; followed by free fall after separation from the stump with constant vertical acceleration and constant angular velocity. This article considers only the first stage, which cannot be described in terms of elementary functions. The second is trivially described mathematically, though is complicated in practice by the need to determine which part of the tree strikes the ground first.

In order to render the problem soluble I assume

- only gravitational forces and contact forces at the stump are significant,

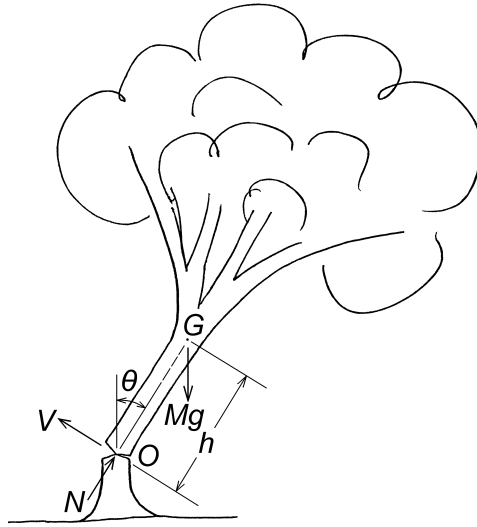


FIGURE 1: Schematic diagram of tree being felled showing relevant algebraic variables

- there is no moment at the point of contact with the stump,
- aerodynamic drag is negligible,
- adequate friction exists at the stump to restrain translation before separation (the cut is jagged), and
- there is no interference to the motion, for example the tree canopy does not contact the ground before separation.

2 Problem definition

In Figure 1, valid until the instant of separation, O is the point of contact between the tree and its stump about which the centre of mass G describes

circular motion with radius h . The tree and the cut are not naturally symmetric so points O and G may be located anywhere with respect to the trunk cross section, but θ is the angle that OG makes with the vertical and is assumed initially to have a finite positive value α . Under the action of gravity θ must increase with time. A contact force exists at O represented by components N and V respectively perpendicular and parallel to OG , and kinematics of circular motion gives the components of the acceleration of G as $h\ddot{\theta}$ and $h\dot{\theta}^2$ also respectively perpendicular and parallel to OG . These directions are therefore the natural choice in which to resolve force components when writing the equations of motion. A gravitational force Mg acts vertically at G . The portion of the tree above the cut has mass M and moment of inertia about point O of I_O . Newton's equations of dynamic equilibrium are described in any university introductory engineering dynamics text such as that by Meriam and Kraige [2], and applied to this problem are

$$I_O\ddot{\theta} = Mgh \sin \theta, \quad (1)$$

$$Mh\ddot{\theta} = Mg \sin \theta - V, \quad (2)$$

$$Mh\dot{\theta}^2 = Mg \cos \theta - N, \quad (3)$$

with initial conditions

$$\theta(0) = \alpha, \quad (4)$$

$$\dot{\theta}(0) = 0. \quad (5)$$

We determine implicitly the time to separation by seeking as follows the time variation of the trunk angle $\theta(t)$ and the angle $\theta = \beta$ at which the tree separates from the stump.

Multiplying (1) by $\dot{\theta}$ and defining the constant

$$A = \sqrt{\frac{I_O}{Mgh}} \quad (6)$$

yields

$$A^2\dot{\theta}\ddot{\theta} = \dot{\theta} \sin \theta. \quad (7)$$

Then integrate (7) with respect to time with initial conditions (4) and (5),

$$\frac{1}{2}A^2\dot{\theta}^2 = \cos \alpha - \cos \theta. \quad (8)$$

Substitute the value of $\dot{\theta}^2$ thus obtained into the axial equilibrium equation (3) at the instant of separation of the tree from the stump, at which $N = 0$, to give the angle of separation as

$$\beta = \arccos \left\{ \frac{\cos \alpha}{1 + I_O/2Mh^2} \right\}. \quad (9)$$

I note that since $\cos \alpha < 1$, $I_O = I_G + Mh^2$ and I_G , the mass moment of inertia about the centre of gravity G , is always positive then $\beta > \arccos 2/3 \simeq 48.2^\circ$. This is an absolute lower bound. More realistic bounds (33) and (34) are discussed in Section 5.

The remaining task is to solve equation (8) between the limits $\theta(0) = \alpha$ and $\theta(t) = \beta$, hence determine t .

3 Analytic solution

I solve differential equation (8) by separation of variables,

$$\frac{A}{\sqrt{2}} \int_{\alpha}^{\beta} \frac{d\theta}{\sqrt{\cos \alpha - \cos \theta}} = \int_0^t dt = t. \quad (10)$$

Now with the change of variable $\theta = \pi - 2\phi$ and defining $\phi_0 = (\pi - \alpha)/2 = \phi$ at $t = 0$,

$$t = \frac{A}{\sqrt{2}} \int_{\frac{\pi-\alpha}{2}}^{\frac{\pi-\beta}{2}} \frac{-2 d\phi}{\sqrt{2}\sqrt{\sin^2 \phi_0 - \sin^2 \phi}}$$

$$\begin{aligned}
 &= \frac{A}{\sin \phi_0} \left\{ \int_0^{\frac{\pi-\alpha}{2}} \frac{d\phi}{\sqrt{1-m \sin^2 \phi}} - \int_0^{\frac{\pi-\beta}{2}} \frac{d\phi}{\sqrt{1-m \sin^2 \phi}} \right\} \\
 &= \frac{A}{\sin \phi_0} \left\{ F\left(\frac{\pi-\alpha}{2} \mid m\right) - F\left(\frac{\pi-\beta}{2} \mid m\right) \right\}, \tag{11}
 \end{aligned}$$

where $m = 1/\sin^2 \phi_0$. Here t is expressed in terms of elliptic integrals of the first kind $F(\varphi \mid m)$ defined by Abramowitz and Stegun [1, §17.2].

It is usual to transform F so that $0 \leq m \leq 1$. I use the result $F(\varphi \mid m) = m^{-1/2} F(\theta \mid m^{-1})$ with $\sin \theta = m^{1/2} \sin \varphi$ [1, §17.4.15]. After some simplification,

$$\begin{aligned}
 t &= A \left\{ F\left(\frac{\pi}{2} \mid \cos^2 \frac{\alpha}{2}\right) - F\left(q_\beta \mid \cos^2 \frac{\alpha}{2}\right) \right\} \\
 &= A \left\{ K\left(\cos^2 \frac{\alpha}{2}\right) - F\left(q_\beta \mid \cos^2 \frac{\alpha}{2}\right) \right\}, \tag{12}
 \end{aligned}$$

where $q_\beta = \arcsin\left(\frac{\cos \beta/2}{\cos \alpha/2}\right)$, A is given by (6) and K is the complete elliptic integrals of the first kind [1, S17.3].

Further reduction using double angle formulas and the expression (9) for β gives

$$q_\beta = \arcsin \sqrt{\frac{1 + \cos \alpha / \left(1 + \frac{I_O}{2Mh^2}\right)}{1 + \cos \alpha}}. \tag{13}$$

Thus the time to separation, normalised by A , is a function of $I_O/2Mh^2$ and α only.

4 Numerical solution

The analytic solution may be concisely coded in programs such as Matlab. However, this work was undertaken for a client from a background without

high level mathematical skills and with access only to spreadsheets to perform computations. A numerical method was sought that

- was practical and easily used by non-mathematicians,
- was easily implemented in a spreadsheet, and
- gave appropriate accuracy bearing in mind the imprecise nature of the input parameters.

Errors of less than 2% were deemed to meet or exceed the last criterion.

Traditional time stepping methods used for ODEs were considered unsuitable because they generally require a large number of time steps to achieve reasonable accuracy, which would quickly clutter a spreadsheet, and they would require interpolation or iteration as they do not in general explicitly solve for t .

In contrast a method based on a truncated Taylor series proved ideal.

First, expand $\theta(t) = \beta$ as a Taylor series where t is time to separation,

$$\beta = \theta(0) + \dot{\theta}(0)t + \ddot{\theta}(0)\frac{t^2}{2} + \dots \quad (14)$$

We have from (1) and (6) that $\ddot{\theta} = \sin\theta/A^2$, giving sufficient information to express arbitrarily high order derivatives of θ in terms of θ and $\dot{\theta}$. Up to sixth order,

$$A^2 \frac{d^3\theta}{dt^3} = \cos\theta \dot{\theta}, \quad (15)$$

$$A^2 \frac{d^4\theta}{dt^4} = \ddot{\theta} \cos\theta - \dot{\theta}^2 \sin\theta, \quad (16)$$

$$A^2 \frac{d^5\theta}{dt^5} = \frac{d^3\theta}{dt^3} \cos\theta - 3\ddot{\theta}\dot{\theta} \sin\theta - \dot{\theta}^3 \cos\theta, \quad (17)$$

$$A^2 \frac{d^6 \theta}{dt^6} = \frac{d^4 \theta}{dt^4} \cos \theta - 4 \frac{d^3 \theta}{dt^3} \dot{\theta} \sin \theta - 6 \ddot{\theta} \dot{\theta}^2 \cos \theta - 3 \dot{\theta}^2 \sin \theta + \dot{\theta}^4 \sin \theta. \quad (18)$$

Putting $\dot{\theta}(0) = 0$ and $\theta(0) = \alpha$ reduces even derivatives at $t = 0$ to

$$\frac{d^4 \theta(0)}{dt^4} = \frac{1}{A^4} \sin \alpha \cos \alpha, \quad (19)$$

$$\frac{d^6 \theta(0)}{dt^6} = \frac{1}{A^6} \sin \alpha (\cos^2 \alpha - 3 \sin^2 \alpha), \quad (20)$$

and odd derivatives vanish. Thus to sixth (or seventh) order

$$\beta = \alpha + \sin \alpha \frac{\tau^2}{2!} + \sin \alpha \cos \alpha \frac{\tau^4}{4!} + \sin \alpha (\cos^2 \alpha - 3 \sin^2 \alpha) \frac{\tau^6}{6!}, \quad (21)$$

where I introduce dimensionless time $\tau = t/A$.

If it is reasonable to assume that the tree does not lean substantially, then the small α approximation may be used:

$$\beta \simeq \alpha \left\{ 1 + \frac{\tau^2}{2!} + \frac{\tau^4}{4!} + (1 - 3\alpha^2) \frac{\tau^6}{6!} \right\}. \quad (22)$$

4.1 Fourth order approximation

To fourth order the problem is described by the quadratic in τ^2

$$a\tau^4 + b\tau^2 + c = 0, \quad (23)$$

where

$$a = \frac{\sin \alpha \cos \alpha}{4!}, \quad (24)$$

$$b = \frac{\sin \alpha}{2!}, \quad (25)$$

$$c = \alpha - \beta. \quad (26)$$

The time to separation may be obtained explicitly as the solution to (23). Since $c < 0$ one of the roots τ^2 of (23) will be negative and not physically meaningful, so the positive root is to be taken. However, the results by this method differed from the analytic solution by up to 10% for typical input parameters, so a higher order approximation was deemed necessary.

4.2 Sixth order approximation

Comparing (21) and (23), sixth order accuracy is obtained by replacing c in (26) with

$$c_i = \alpha - \beta + \frac{\tau_i^6}{6!} \sin \alpha (\cos^2 \alpha - 3 \sin^2 \alpha). \quad (27)$$

Solving the quadratic in τ^2 (23),

$$\tau_{i+1} = \sqrt{\frac{6}{\cos \alpha} \left(\sqrt{1 - \frac{2c_i}{3} \cot \alpha} - 1 \right)^{1/2}}. \quad (28)$$

In this case it is necessary to iterate between (27) and (28) to obtain successive approximations τ_i . The initial approximation $\tau_1 = 0$ was used and 2–3 iterations were sufficient to confine errors to within 2%, requiring only 4–6 additional lines in the spreadsheet.

The sixth order small α approximation, after cancelling common factor α , is

$$a = \frac{1}{4!}, \quad (29)$$

$$b = \frac{1}{2!}, \quad (30)$$

$$c = 1 - \frac{\beta}{\alpha} + \frac{\tau^6}{6!} (1 - 3\alpha^2), \quad (31)$$

from which

$$\tau = \sqrt{6} \left(\sqrt{1 - \frac{2c}{3}} - 1 \right)^{1/2}. \quad (32)$$

Three sources of error are present in this numerical method arising from

1. truncation of the Taylor series—sixth order terms were necessary and in most cases sufficient to meet the accuracy criterion;
2. termination of the iteration between (31) and (32)—this error is easily controlled, but 2–3 iterations were found sufficient to meet the accuracy criterion, and underrelaxation by averaging the new and old τ accelerated convergence;
3. the small α approximation (if used)—this proved of no concern for $\alpha < 15^\circ$.

Error results presented in Figure 4 are discussed in more detail below.

5 Results

Figure 2 shows theoretical dimensionless felling times computed analytically by (12) for various initial and separation angles. Each value of β implies a certain I_0/Mh^2 , not all of which are possible as noted in section 2. To identify likely values of β consider I_0/Mh^2 in the denominator of (9) and recall that $I_O = I_G + Mh^2$. A realistic but still conservative lower bound for I_G is obtained by assuming the tree to approximate a uniform slender rod of length $2h$, thus $I_G \geq Mh^2/3$ and from (9)

$$\beta \geq \arccos \left(\frac{\cos \alpha}{5/3} \right) \geq 53.13^\circ. \quad (33)$$

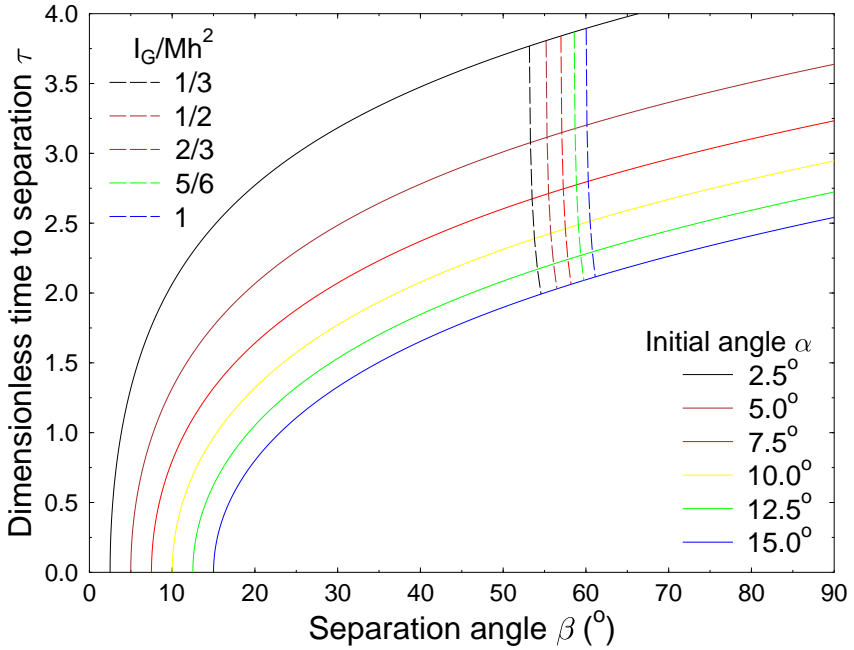


FIGURE 2: Felling time as a function of β : Solid lines, constant α ; Dashed lines, constant I_G/Mh^2 .

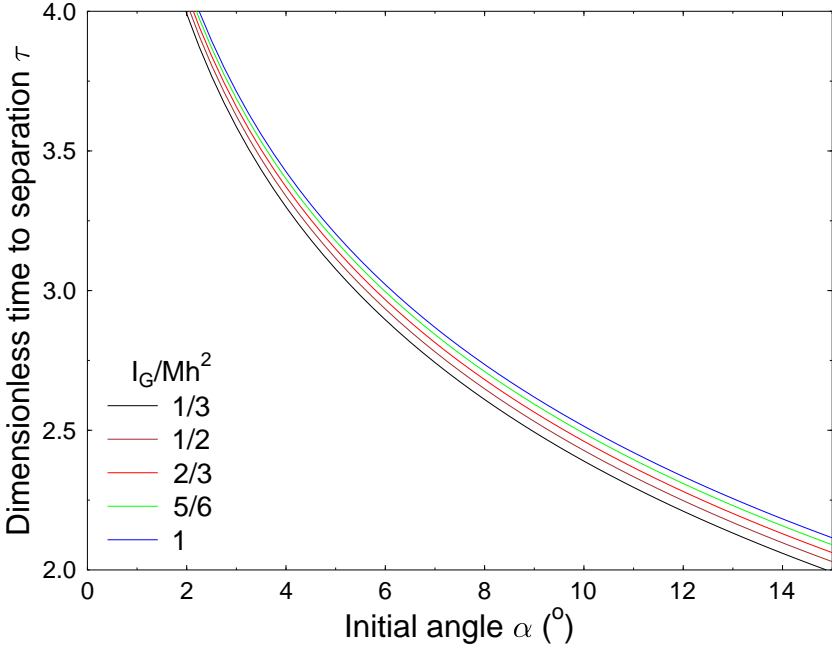


FIGURE 3: Felling times as a function of initial angle α for various I_G/Mh^2 .

If the tree is approximated as a uniform sphere of radius H , $I_G = \frac{2}{5}Mh^2$ and $\beta \geq \arccos(\cos \alpha/1.7)$. However, as a practical upper bound for β I assume $\alpha = 15^\circ$ and a denominator of 2 in (9), corresponding to $I_G = Mh^2$, giving

$$\beta \leq \arccos\left(\frac{\cos 15^\circ}{2}\right) \simeq 61.12^\circ. \tag{34}$$

Values within these limits are shown in Figure 2 by the dashed lines, representing contours of constant I_G/Mh^2 within the above range. Note that quite a narrow range of β results.

Figure 3 shows felling time as a function of initial angle and I_G/Mh^2 , which together are sufficient to uniquely determine it. We see perhaps sur-

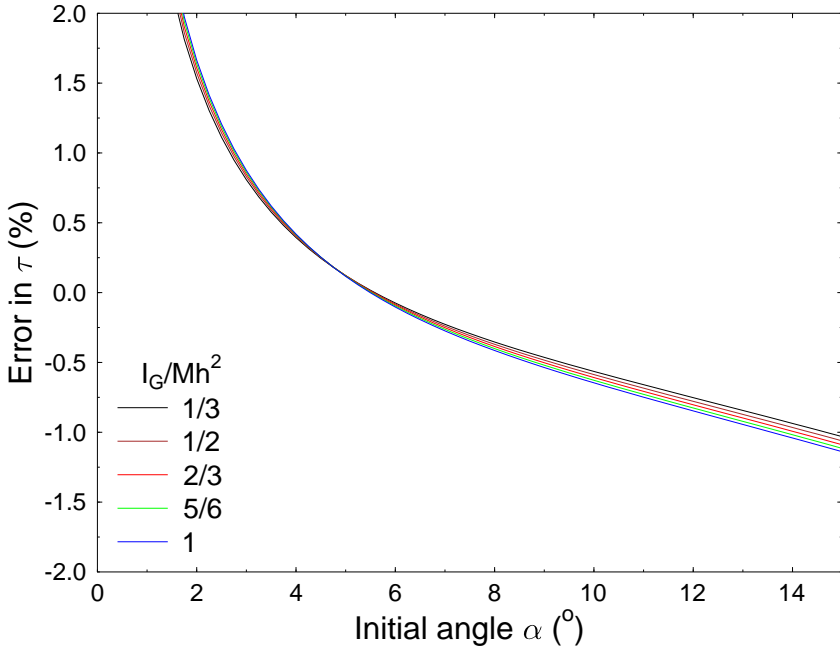


FIGURE 4: Error in computed felling times using the 6th order small α approximation.

prisingly that the tree's moment of inertia has relatively little influence on the felling time, while as expected the felling time is strongly dependent on the initial lean angle α .

Figure 4 shows that error in the sixth order small α approximation is predominantly a function of α . For small α , τ increases rapidly and the error therefore is dominated by the truncation of the Taylor series. The error can be reduced if necessary by inclusion of higher order terms. For large α the error is dominated by the small α approximation. This component can be eliminated entirely by using the finite α form in which (27) and (28) replace (31) and (32).

6 Conclusion

The numerical method of Section 4 proved a simple and accurate means to determine the time taken for a cut tree to fall to its point of separation from the stump. In this method the time

$$t = A\tau(\alpha, I_O/Mh^2) \quad (35)$$

is fully specified by equations (6), (9), (31) and (32).

The method achieved the desired objectives that

- it does not require high level mathematical skills—it can be understood using first year undergraduate mathematics and implemented with much less;
- it is easily programmed in a spreadsheet;
- it introduces errors of less than 2% over a realistic range of inputs, which is considerable less than the uncertainty in the input parameters.

References

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Author address

1. **D. S. Holloway**, School of Engineering, University of Tasmania, Hobart, AUSTRALIA.
<mailto:Damien.Holloway@utas.edu.au>