Numerical solution to the Saffman–Taylor finger problem with kinetic undercooling regularisation

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Abstract

The Saffman–Taylor finger problem is to predict the shape and, in particular, width of a finger of fluid travelling in a Hele–Shaw cell filled with a different, more viscous fluid. In experiments the width is dependent on the speed of propagation of the finger, tending to half the total cell width as the speed increases. To predict this result mathematically, nonlinear effects on the fluid interface must be considered; usually surface tension is included for this purpose. This makes the mathematical problem sufficiently difficult that asymptotic or numerical methods must be used. We adapt numerical methods used to solve the Saffman–Taylor finger problem with surface tension to instead include the effect of kinetic undercooling, a regularisation effect important in Stefan melting-freezing problems, for which Hele–Shaw flow serves as a leading order approximation when the specific heat of a substance is much smaller than its latent heat. We find the

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existence of a solution branch where the finger width tends to zero as the propagation speed increases, disagreeing with some aspects of the asymptotic analysis of the same problem. We also find a second solution branch, supporting the idea of a countably infinite number of branches as for the surface tension problem.

Contents

1 Introduction

In highly viscous flow, when a fluid of a certain viscosity is used to push a more viscous fluid, the interface between the two fluids is unstable. This phenomenon is known to mathematicians as the Saffman–Taylor instability, after the pioneering experimental and analytical study of Saffman and Taylor into the effect in the Hele–Shaw cell; an experimental device consisting of two

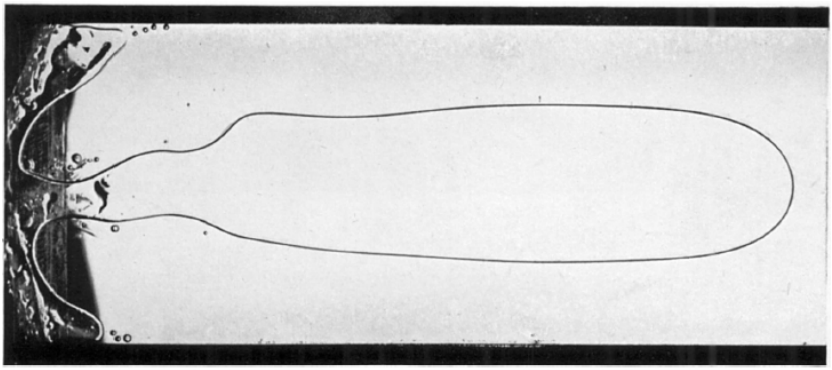

FIGURE 1: A Saffman–Taylor finger consisting of air injected into a Hele–Shaw cell filled with glycerin [1, Fig. 8] (reproduced with permission of the publisher).

narrowly separated glass plates with the fluids sandwiched in between, which effectively reduces the fluid flow problem to two dimensions [1].

Saffman and Taylor constructed a long, 'narrow channel' Hele–Shaw cell, where fluid could be injected or removed at either end. By injection of the less viscous driving fluid at one end, the instability leads to the formation of long thin 'fingers' of the less viscous fluid, with a single finger eventually dominating over the rest (see Figure 1). Saffman and Taylor observed that the shape of the finger depends on the velocity of that finger. In particular, as they were driven faster, the fingers' widths tended to half that of the total cell. In other words, letting $\lambda \in [0, 1]$ be the ratio between finger width and cell width, $\lambda \to 1/2^+$ as speed increases. This posed an interesting mathematical problem: how can we determine the finger shape and width as a function of the speed?

The answer to this problem lies in the boundary condition we apply on the interface between the two fluids. Saffman and Taylor used Bernoulli's condition, in which the pressure along the interface is constant. This results in a linear boundary value problem which is solvable in closed form, but has a

continuous family of solutions for any finger width λ and so does not predict the width at all. Boundary conditions that do select for λ, for instance by including the effects of surface tension, are nonlinear and are characterised by a dimensionless parameter (such as a surface tension parameter σ) that goes to zero as the finger speed goes to infinity. Problems with these boundary conditions must be analysed using numerical or asymptotic methods, as exact solutions can no longer be found using linear solution techniques.

McLean and Saffman [2] treated the Saffman–Taylor finger problem with surface tension on the free boundary using complex variable techniques to reduce the problem to a coupled nonlinear integro-differential system of equations. This system was solved numerically by discretising the independent variable at a set of node points and approximating derivatives and integrals by finite difference formulae and the trapezoid rule, respectively. This numerical approach subsequently showed that there are a countable infinity of solutions with different widths for each value of the surface tension [3]. All these solution branches have $\lambda \to 1/2^+$ as the surface tension parameter $\sigma \to 0$. The asymptotic behaviour for small surface tension (equivalently, large finger speed) has also been studied extensively and makes the same prediction, although a beyond-all-orders (or exponential) asymptotic analysis is needed to distinguish the discrete values of λ for which the solution exists [4, 5, 6].

Here we examine the less studied kinetic undercooling type boundary effect. Kinetic undercooling is a condition of physical meaning in Stefan melting-freezing problems, where the melting temperature is not constant but instead weakly dependent on the speed of the moving front between the two phases. The equations for Hele–Shaw flow can be considered a leading order approximation to the Stefan problem for small specific heat, and in this context the kinetic undercooling condition translates to a relationship between the pressure on the interface and the normal interface velocity [7]. This problem is characterised by a nondimensional kinetic undercooling parameter ϵ which, as with σ for surface tension, goes to zero as finger speed increases.

Chapman and King [8] undertook the beyond-all-orders asymptotic analysis

of the Saffman–Taylor finger with kinetic undercooling. The details, which are similar to, but more difficult than, the surface tension case [4, 5, 6], provide an asymptotic approximation valid for $\epsilon \ll 1$ with $\lambda - 1/2 = \mathcal{O}(1)$ (that is, for a given ϵ, solutions on sufficiently high branches so that λ is not yet close to $1/2$). Further analysis for $\lambda - 1/2 = \mathcal{O}(\epsilon^{2/3})$ predicts that $\lambda \to 1/2^+$ as $\epsilon \to 0$, though the explicit form of the asymptotic relationship between ϵ and λ is intractable in this limit.

We adapted the numerical discretisation scheme of McLean and Saffman [2] to include kinetic undercooling instead of surface tension. Our major conclusion is that a solution branch exists for which λ decreases to less than $1/2$ for some ϵ much greater than zero, which appears to be at odds with some aspects of the asymptotic analysis. In particular, it appears that $\lambda \to 0^+$ as $\epsilon \to 0$ on the lowest solution branch. The existence of an infinite number of solution branches is suggested, but not confirmed.

2 Boundary integral formulation

Hele–Shaw flows are mathematically attractive since the fluid flow is effectively two dimensional, and the pressure acts as a velocity potential, satisfying Laplace's equation, meaning complex variable techniques can be used. We formulate the differential-integral system in the same manner as McLean and Saffman [2] and Chapman and King [8], to whom we refer for more detail.

Consider the problem where a less viscous fluid (taken to be inviscid) is injected into the left hand side of a Hele–Shaw cell with channel walls at $y = \pm 1$, initially filled with the more viscous fluid. Once the viscous finger has developed and is moving at constant speed, we may set the frame of reference so that the finger is steady. The driving inviscid fluid means the viscous fluid has a nonzero velocity in the far field (as $x \to \infty$), which depends on the speed of the finger's propagation.

Given that the velocity potential ϕ (proportional to the pressure) satisfies

Laplace's equation, the velocity field is described in terms of either ϕ or the streamfunction ψ by

$$u = \phi_x = \psi_y\,, \quad v = \phi_y = -\psi_x\,,$$

where subscripts denote partial derivatives. If the finger velocity is U, and the far field velocity in the stationary frame of reference is V, the far field conditions are

$$\lim_{x\to-\infty} \psi_y(x,y) = -U\,, \quad \lim_{x\to\infty} \psi_y(x,y) = V - U\,.$$

Let the finger width be λ and $(x,y) \in \Gamma$ be the coordinates of the free boundary of the finger. Both the free boundary and channel walls are streamlines, so taking $\psi = 0$ on the centreline $x = 0$ (and therefore also on Γ),

$$\psi(x,\pm 1) = \pm(V-U) \Rightarrow \psi(-\infty,y) = V - Uy\,,$$

and since $\psi(-\infty,\lambda) = 0$ we must have $\lambda = V/U$. It is convenient to scale time such that $V - U$ (the velocity as $x \to \infty$) is unity. In this case

$$U = \frac{1}{1-\lambda}\,, \quad V = 1 + \frac{1}{1-\lambda}\,.$$

2.1 Conformal mapping

Instead of attempting to apply some boundary condition on the free surface Γ, whose coordinates are unknown, we use a change of variables to map the free surface to a fixed line segment. Let $f = \phi + i\psi$ be the complex potential, and define the complex variable

$$\zeta = \xi + i\eta = e^{\pi f}.$$

Assuming the finger is symmetric about the centreline, this maps one half of the fluid domain outside the finger to the upper half ζ-plane, and the free surface (letting $\phi = 0$ at the nose of the finger) to the real line segment $\xi \in [0,1]$, with $\xi = 1$ the nose of the finger, and $\xi = 0$ the tail as $x \to -\infty$ (see Figure 2).

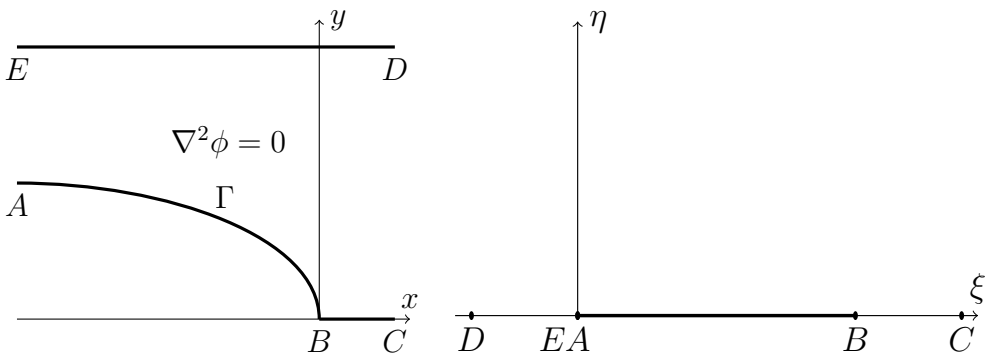

FIGURE 2: A schematic of the fluid domain in the physical plane, and the conformal mapping to the $\zeta = e^{\pi f}$ plane. The letters A to E mark corresponding points in each plane. The interface Γ maps to the line segment $\xi \in [0, 1]$.

2.2 Integral equation

The complex velocity

$$w(z) = \frac{d\phi}{dz} = \hat{q}e^{-i\hat{\theta}},$$

where $\hat{q}$ and $\hat{\theta}$ are the speed and velocity angle. Since $\log(-w) = \ln \hat{q} - i(\hat{\theta} - \pi)$ is an analytic function, its real and imaginary parts are related by the Hilbert transform integral

$$\ln \hat{q} = -\frac{1}{\pi} \fint_0^1 \frac{\hat{\theta}(\xi') - \pi}{\xi' - \xi} \, d\xi', \tag{1}$$

since $\hat{\theta} = \pi$ everywhere on the ξ-axis except in $[0, 1]$. The dashed integral sign indicates it is of Cauchy principal value type. Note that we need $\hat{q} \to 1$ as $\phi \to -\infty$ (or $x \to \infty$) from the previous section, so that $\log(-w)$ vanishes in the far field of the upper half plane.

2.3 Differential equation

In the Hele–Shaw context, the kinetic undercooling assumption is that the pressure on the moving boundary is proportional to its normal velocity v_n [8]. In the moving frame of reference this translates to

$$\phi = c v_n - \frac{x}{1-\lambda}\,, \tag{2}$$

where c is the kinetic undercooling coefficient. Let the finger velocity $U = 1/(1-\lambda)$, so that $v_n = \sin\hat{\theta}\,/(1-\lambda)$. Defining s to be the arc length parameter for the free boundary,

$$\frac{d\phi}{ds} = \hat{q}\,, \quad \frac{dx}{ds} = \cos\hat{\theta}\,, \quad \frac{d}{ds} = \frac{d\xi}{d\phi}\frac{d\phi}{ds}\frac{d}{d\xi} = -\pi\hat{q}\xi\frac{d}{d\xi}\,.$$

Differentiating (2) with respect to s results in the differential equation

$$\hat{q} = -\frac{c\pi}{1-\lambda}\hat{q}\cos\hat{\theta}\xi\frac{d\hat{\theta}}{d\xi} - \frac{1}{1-\lambda}\cos\hat{\theta}\,. \tag{3}$$

Lastly, we simplify (1) and (3) by defining $q = (1-\lambda)\hat{q}$ and $\theta = \hat{\theta} - \pi$. The integro-differential system, with associated boundary conditions, becomes

$$\begin{aligned}
&2\epsilon q\cos\theta\xi\frac{d\theta}{d\xi} + \cos\theta - q = 0\,,\\
&\ln q = \ln(1-\lambda) - \frac{1}{\pi}\fint_0^1 \frac{\theta(\xi')}{\xi'-\xi}\,d\xi'\,,\\
&\ln(1-\lambda) = \frac{1}{\pi}\fint_0^1 \frac{\theta(\xi)}{\xi}\,d\xi\,,\\
&\theta(0) = 0\,, \quad \theta(1) = -\frac{\pi}{2}\,,
\end{aligned} \tag{4}$$

where $\xi \in [0,1]$, and

$$\epsilon = \frac{c\pi}{2(1-\lambda)} \tag{5}$$

is the new kinetic undercooling parameter.

2.4 Cauchy principal value and endpoint singularities

Before we proceed to the numerical solution method there are two issues that must be dealt with. Firstly, a Cauchy principal value integral is difficult to deal with numerically, so we subtract out the singular part:

$$\fint_0^1 \frac{\theta(\xi')}{\xi'-\xi}\,d\xi' = \int_0^1 \frac{\theta(\xi')-\theta(\xi)}{\xi'-\xi}\,d\xi' + \theta(\xi)\ln\left(\frac{1-\xi}{\xi}\right).$$

Secondly, McLean and Saffman [2] noted that in ξ, both θ and q are undifferentiable at both endpoints. At $\xi = 1$ a square root singularity exists due to the effect of the stagnation point on the conformal mapping. At $\xi = 0$ the power of the singularity α is obtained from assuming local expansions $\theta \sim a\xi^\alpha$ and $q \sim 1 + b\xi^\alpha$. Substitution into the system (4) gives

$$2\epsilon\alpha a\xi^\alpha = -b\xi^\alpha + \mathcal{O}(s^{2\alpha}), \quad b\xi^\alpha = a\cot(\pi\alpha)\xi^\alpha + \mathcal{O}(s^{2\alpha}),$$

so α is found numerically as the smallest root of the equation $2\epsilon\alpha + \cot(\pi\alpha) = 0$. We now remove the endpoint singularities by the change of variable $t = \sqrt{1-\xi^\alpha}$, resulting in

$$-\alpha\epsilon\cos\theta\, q\frac{1-t^2}{t}\frac{d\theta}{dt} + \cos\theta - q = 0, \tag{6}$$

and

$$q = (1-\lambda)\left[(1-t^2)^{-1/\alpha}-1\right]^{-\theta/\pi} \times \exp\left\{-\frac{2}{\alpha\pi}\int_0^1 \frac{t'}{(1-t'^2)^{1-1/\alpha}}\frac{\theta(t')-\theta(t)}{(1-t'^2)^{1/\alpha}-(1-t^2)^{1/\alpha}}\,dt'\right\}. \tag{7}$$

3 Numerical method

To solve the coupled differential-integral system (4) we discretise the domain $[0,1]$ into $N+1$ evenly spaced node points

$$t_j = j\Delta t, \quad j = 0,\ldots,N,$$

where $\Delta t = 1/N$, and seek $\theta_j \approx \theta(t_j)$ and $q_j \approx q(t_j)$. For a given set of θ_j values, we compute q_j by numerical integration in (7), and the discretisation of the differential equation (6) results in a nonlinear system of equations for θ_j. We use Simpson's rule for integration, and the third order finite difference formula

$$\Delta\theta_1 = \frac{\theta_0}{3} - \frac{\theta_1}{2} + \theta_2 = \frac{\theta_3}{6} \quad \text{and} \quad \Delta\theta_j = \frac{\theta_{j-2}}{6} - \theta_{j-1} + \frac{\theta_j}{2} + \frac{\theta_{j+1}}{3},$$

for $j = 2, \ldots, N-1$, for the derivative in (6), as well as for dealing with the removable singularity in (7). The nonlinear system for the θ_i values was solved using Newton's method, using a finite difference approximation to compute the Jacobian.

4 Results

The scheme outlined above was programmed in Matlab and run on a desktop computer. For $N = 50$, which produced sufficiently precise results for most values of ϵ, the code took about 0.1 seconds to converge for each value of ϵ.

To observe a single branch of the relation between kinetic undercooling parameter ϵ and finger width λ, we solved the nonlinear system starting with $\epsilon = 1$ using the Saffman–Taylor ($\epsilon = 0$, $\lambda = 1/2$) solution as an initial guess. The kinetic undercooling parameter ϵ was then decreased gradually to 0.001, using the previous solution as the initial guess at each value of ϵ. The resulting dependence of λ on ϵ is provided as the lower line in Figure 3. On this branch λ goes below $1/2$ for finite λ, with $\lambda \to 0^+$ as $\epsilon \to 0$.

We also found a second solution branch by starting at $\epsilon = 0.1$ with the Saffman–Taylor solution as initial guess and then increasing ϵ. Figure 3 plots this solution branch, again using $N = 50$.

As a numerical check we tested the convergence properties of the method as N increases. While both integral and differential approximations are

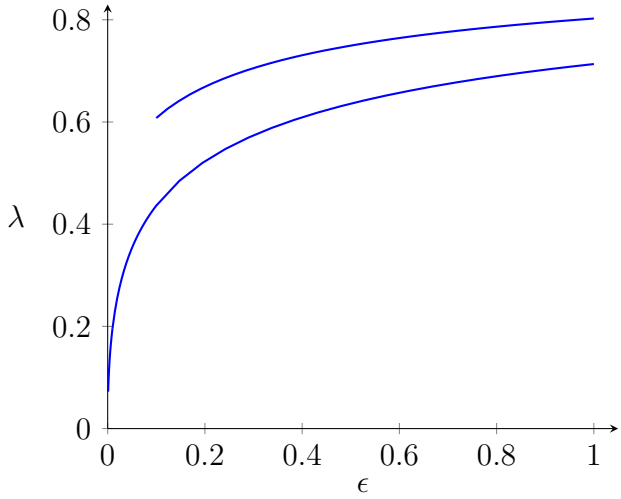

FIGURE 3: Two solution branches for finger width λ as a function of kinetic undercooling parameter ϵ. The lower solution branch has $\lambda \to 0^+$ as $\epsilon \to 0$.

theoretically third order, we found that the rate of convergence is dependent on ϵ. For relatively large values (say $\epsilon \approx 1$), 50 node points was sufficient for convergence to three decimal places. However, for small values convergence was much slower. Figure 4 shows the change in λ over N for $\epsilon = 0.05$. This figure also shows that the convergence was well behaved and decreasing, which suggests that the existence of solutions with $\lambda < 1/2$ is not a numerical artefact.

5 Discussion

We successfully adapted the numerical scheme of McLean and Saffman [2] to the problem of viscous fingers with a kinetic undercooling condition on the interface. The results predict the existence of a solution branch which has $\lambda \to 0^+$ as $\epsilon \to 0$. This is unexpected, given the asymptotic analysis [8],

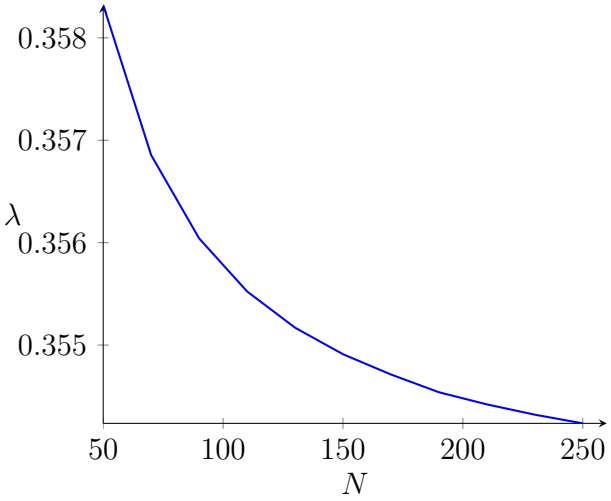

FIGURE 4: The value of λ computed for $\epsilon = 0.05$ for increasing number of node points N. Convergence is slow but well behaved and decreasing, suggesting the existence of solutions with $\lambda < 1/2$ is not due to numerical error.

which has $\lambda - 1/2 = \mathcal{O}(\epsilon^{2/3})$ as $\epsilon \to 0$.

Our numerical work has also shown the existence of a second solution branch. The asymptotics predicts the existence of a countable infinity of solution branches, as with the surface tension problem. The existence of these branches was demonstrated numerically for the surface tension problem [3]. While we predict that a careful application of the same method to the kinetic undercooling problem will confirm the existence of countably many solution branches as well, this remains to be done systematically.

In order to fully understand the results of our study, it would be worth performing a stability analysis on the problem. In the corresponding problem, for which there is surface tension acting on the interface of the finger, the lowest solution branch is linearly stable, while all others are unstable [9, 10, 11], explaining why the lowest solution branch is that seen in experiments. For

our case, in which surface tension is replaced by kinetic undercooling, an analogous analysis would presumably indicate which branches are stable.

Similarly, a worthwhile extension would be to treat the time dependent problem of an almost flat interface advancing along a long narrow channel, eventually forming a single longer finger (that would become the travelling wave finger in the long time limit). In this case the corresponding problem with surface tension has been extensively examined [12, 13, 14, 15] and confirms that ultimately a single finger emerges with the value of λ predicted by the Mclean–Saffman solution (except for very small surface tension, where tip splitting occurs). With surface tension replaced by kinetic undercooling, it is not at all obvious what form the numerical solutions will take, and whether kinetic undercooling has the effect of selecting the finger with $\lambda = 1/2$ (perhaps corresponding to solutions on the upper curve in Figure 3) or a long needle with $\lambda \ll 1$ for $\epsilon \ll 1$ (corresponding to the lower curve in Figure 3).

Finally, as we already noted, the original experiments of Saffman and Taylor [1] and many others show that the physically relevant ratio for the selection problem is $\lambda = 1/2$, not $\lambda = 0$. Thus we speculate that the proposed stability and time dependent analysis above will show that the lower branch in Figure 3 is not seen in experiments. We conclude that great care must be taken when applying the kinetic undercooling regularisation to Hele–Shaw problems, with surface tension being the preferred mechanism.

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