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CVaR-minimising hedging by a smoothing method

T. Tarnopolskaya[1] Z. Zhu[2]

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Abstract

Minimisation of a conditional value-at-risk (CVaR) is a non-smooth stochastic minimisation problem with the non-smoothness caused by a plus function in the integrand of the objective function. We study the performance of several smoothing approximations of the plus function in the CVaR minimisation, using an example of a one period CVaR-minimising hedging. The smooth plus function that outperforms others is identified. The convergence of the solution of the smoothed CVaR minimisation problem with increase in the sample size and with decrease in the smoothing parameter is illustrated. The performance of the one period CVaR-minimising hedging and delta-gamma hedging are compared in terms of several commonly used performance criteria. Numerical simulations show that the magnitude and the probability of large losses are smaller in the CVaR-minimising hedge, compared to the delta-gamma hedge. This often occurs at the expense of a deteriorated expected return and increased variance of the hedge. We

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identify the situations when the CVaR-minimising hedge outperforms the delta-gamma hedge according to all performance criteria.

Contents

1 Introduction

There is strong emphasis in the financial industry on managing large unexpected losses. While value-at-risk (VaR) is an international benchmark in the finance industry, CVaR is gaining popularity as the portfolio and the hedging risk measure [1, 2, 7, 10, 11, 13, 14, 15], as it rectifies several shortcomings of VaR (for example, lack of sub-additivity and convexity [10, 11]), while preserving its useful intuitive meaning (section 2).

CVaR minimisation is a non-smooth stochastic minimisation problem with the non-smoothness caused by a plus function in the integrand of the objective function [10, 11]. In practice, the expected value in the expression for CVaR is usually approximated with the sample average. In such a setting, a CVaR minimisation problem reduces to a linear programming problem [10, 11]. However, such a problem becomes prohibitively expensive when the sample size or the number of instruments in the portfolio is large. While some numerical methods have been developed for solving non-smooth stochastic optimisation problems, a simple smoothing technique to deal with the non-smoothness of a stochastic program in CVaR has recently been proposed [1, 2, 16] and showed promising results.

This article continues this line of research. Section 3 reviews several smooth approximations for the plus function available and show that some of them are identical, while others can be transformed into each other using a simple change of a smoothing parameter. We select three distinctive smooth plus functions and implement them in CVaR-minimising algorithms, using an example of one period CVaR-minimising hedging. We compare their performance and convergence against the traditional linear programming algorithm. A smooth plus function that outperforms the others is identified. The convergence of the solution of the smoothed CVaR minimisation problem, with increase in the sample size and with decrease in the smoothing parameter, is discussed and illustrated.

Portfolio hedging aims at reducing the risk of an investment (target portfolio) by making an offsetting investment (hedging portfolio). While different types of hedging exist, one commonly used approach is to match the sensitivities of the target and hedging portfolios (so-called sensitivity hedging such as delta hedging, delta-gamma hedging and others). Sensitivity hedges are instantaneously risk-free if they are re-balanced continuously. However, this balancing is not possible in practice and the sensitivity hedges are implemented at discrete time intervals. As a result, the hedged portfolio is no longer risk-free. If possible, static hedging (replication) should always be relied on to remove or reduce the risk. However, derivatives contracts generally cannot be readily hedged statically. A trading book of derivatives is normally hedged dynamically via the traditional delta-gamma hedging. An alternative strategy may be to minimise a particular measure of risk (for example, CVaR) for a chosen time horizon. A smoothing method is shown to speed up the CVaR minimisation significantly and makes CVaR-minimising hedging a viable alternative to delta-gamma hedging in practice. We study the benefits of CVaR-minimising hedging against the traditional dynamic delta-gamma hedging.

We compare the performance of the one period CVaR-minimising and the delta-gamma hedging, in terms of several commonly used performance criteria. Numerical simulations in section 4 show that the main advantage of the CVaR-minimising hedging over the delta-gamma hedging is the reduction in both the magnitude and the probability of large losses in the portfolio. The numerical simulations show that this reduction is often achieved at the expense of a deteriorated expected profit and an increased variance of the portfolio loss distribution. We identify situations when the CVaR-minimising hedge outperforms the delta-gamma hedge according to several commonly used performance criteria.

2 CVaR minimisation

2.1 CVaR as a risk measure

Let $f(x, s)$ be a loss function associated with the decision vector x and a random vector s with a probability density function $\rho(s)$ that represents market uncertainties. For a given portfolio, let $\Psi(x, \beta)$ denote the probability of the loss function $f(x, s)$ not exceeding a particular value β:

$$\Psi(x, \beta) = \int_{f(x,\zeta)\leqslant\beta} \rho(\zeta)\, d\zeta\,.$$

The value-at-risk (VaR) with a confidence level α is defined as $\beta_\alpha(x) = \inf\{\beta \in R : \Psi(x,\beta) \geqslant \alpha\}$, and the conditional value-at-risk (CVaR) is defined as

$$\phi_\alpha(x) = \inf_{\beta\in R}\left\{\beta + \frac{1}{1-\alpha}E[[f(x,s)-\beta]^+]\right\}, \quad (1)$$

where $E[\cdot]$ stands for the expectation and, for any $t \in R$, $[t]^+ = \max\{t, 0\}$.

Rockafellar and Uryasev [10, 11] established that the problem of minimisation of CVaR is equivalent to the problem

$$\min_{x\in R^N}\phi_\alpha(x) = \min_{(x,\beta)\in R^N\times R} F_\alpha(x,\beta), \quad F_\alpha(x,\beta) = \beta + \frac{1}{1-\alpha}E\left[[f(x,s)-\beta]^+\right]. \quad (2)$$

In financial applications, the expected value typically cannot be calculated in a closed form, and is approximated by a sample average through, for example, Monte Carlo simulations. In this case, the problem (2) is replaced by the problem

$$\min_{(x,\beta)\in R^N\times R} F_\alpha^M(x,\beta), \qquad F_\alpha^M(x,\beta) = \beta + \frac{1}{M(1-\alpha)}\sum_{j=1}^{M}[f(x,s^j)-\beta]^+, \quad (3)$$

where the loss function $f(x,s)$ is linear in the decision variables $f(x,s) = x^T v(s) + f_0(s)$, $v(s)$ is the random vector of changes in the prices of portfolio instruments, and $f_0(s)$ represents the loss of the target portfolio. Thus the problem of minimising the CVaR (3) is equivalent to a linear programming (LP) problem.

When the loss function is represented by M scenarios $f(x,s^j)$, $j = 1,\ldots,M$, CVaR is calculated as [10, 11]

$$\phi_\alpha(x) = \frac{1}{1-\alpha}\left[(k_\alpha/M-\alpha)\beta_\alpha(x) + \frac{1}{M}\sum_{k=k_\alpha+1}^{M} f^s(x,s^k)\right], \quad (4)$$

where $f^s(x, s^j)$ are losses sorted in ascending order, and k_α is the unique index satisfying the inequality

$$M - k_\alpha \leqslant (1-\alpha)M < M - k_\alpha + 1 . \tag{5}$$

2.2 CVaR minimisation by a smoothing method

The CVaR minimisation problem (3) is a non-smooth stochastic optimisation problem, with the non-smoothness caused by a plus function in the integrand of the objective function. It can be treated by applying smoothing approximation to the plus function. Then, a smoothed approximate minimisation problem has a form

$$\min_{(x,\beta)\in R^N\times R} \hat{F}_\alpha^M(x,\beta), \quad \hat{F}_\alpha^M(x,\beta) = \beta + \frac{1}{M(1-\alpha)} \sum_{j=1}^{M} \left[\hat{p}(f(x,s^j) - \beta, \epsilon)\right], \tag{6}$$

where $\hat{p}(x, \epsilon)$ is a smooth approximation for a plus function, ϵ is the smoothing parameter. Xu and Zhang [16] recently studied the convergence of stationary points of the approximate problem (6) to those of the original problem (3).

3 Smooth approximations for the plus function

We consider smooth approximations for the plus function $[x]^+$ that satisfy a general definition [16].

Definition 1 *Function* $\hat{p}(x, \epsilon) : R^N \times R \to R$ *(where* ϵ *is a smoothing parameter) is a smoothing of* $[x]^+$ *if it satisfies*

1. $\hat{p}(x, 0) = [x]^+$ *for every* $x \in R^N$*;*
2. $\hat{p}(x, 0)$ *is continuously differentiable on* $R^N \times R\backslash\{0\}$*;*

3. *$\hat{p}(x, 0)$ is locally Lipschitz continuous at $(x, 0)$.*

Many smooth approximations for the plus function $[x]^+$ have been proposed [5, 16]. Below, we review several commonly used smooth approximations.

1. The Neural Networks smooth plus function [4, 5]

$$\hat{p}_1(x,\epsilon) = x + \epsilon \log(1 + e^{-x/\epsilon}). \tag{7}$$

2. Alexander–Coleman–Li smooth plus function [1, 2]

$$\hat{p}_2(x,\epsilon) = \begin{cases} x & \text{if } x > \epsilon\,, \\ (x+\epsilon)^2/(4\epsilon) & \text{if } |x| \leqslant \epsilon\,, \\ 0 & \text{if } x < -\epsilon\,. \end{cases} \tag{8}$$

3. Peng smooth plus function [8]

$$\hat{p}_3(x,\epsilon) = \epsilon \log(1 + e^{x/\epsilon}), \quad \epsilon > 0\,. \tag{9}$$

4. Pinar–Zenios smooth plus function [9]

$$\hat{p}_4(x,\epsilon) = \begin{cases} x - \epsilon/2 & \text{if } x > \epsilon\,, \\ x^2/(2\epsilon) & \text{if } 0 \leqslant x \leqslant \epsilon\,, \\ 0 & \text{if } x < 0\,. \end{cases} \tag{10}$$

5. Chen–Harker, Kanzow, Smale smooth plus function [3, 6, 12]

$$\hat{p}_5(x,\epsilon) = \left(x + \sqrt{x^2 + 4\epsilon^2}\right)/2\,. \tag{11}$$

6. Zang smooth plus function [18]

$$\hat{p}_6(x,\epsilon) = \begin{cases} x & \text{if } x > \epsilon/2\,, \\ (x+\epsilon/2)^2/(2\epsilon) & \text{if } |x| \leqslant \epsilon/2\,, \\ 0 & \text{if } x < -\epsilon/2\,. \end{cases} \tag{12}$$

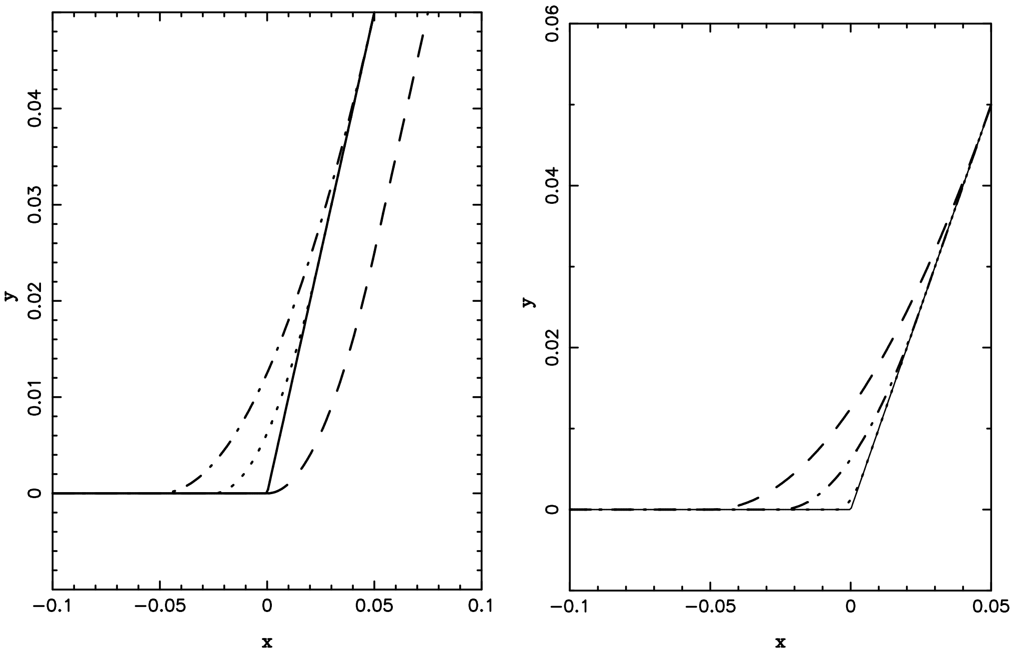

Figure 1: Smooth approximations to the plus function for $\epsilon = 0.05$; the plus function is shown with solid curve; the left figure shows $\hat{p}_4(x,\epsilon)$ (dashed curve), $\hat{p}_6(x,\epsilon)$ (dotted curve) and $\hat{p}_2(x,\epsilon)$ (dashed-dotted curve); the right figure shows $\hat{p}_5(x,\epsilon)$ (dashed curve), $\hat{p}_6(x,\epsilon)$ (dotted curve) and $\hat{p}_1(x,\epsilon)$ ($\hat{p}_3(x,\epsilon)$) (dashed-dotted curve).

One can see that $\hat{p}_1(x,\epsilon)$ is identical to $\hat{p}_3(x,\epsilon)$

$$\begin{aligned}\hat{p}_1(x,\epsilon) &= x + \epsilon\log(1+e^{-x/\epsilon}) = x + \epsilon\log((1+e^{x/\epsilon})e^{-x/\epsilon}) \\ &= x + \epsilon\log(1+e^{x/\epsilon}) - x = \hat{p}_3(x,\epsilon).\end{aligned}$$

With the smoothing parameter change $\epsilon = \tilde{\epsilon}/2$, then $\hat{p}_2(x,\epsilon)$ transforms into $\hat{p}_6(x,\tilde{\epsilon})$, which, in turn, then transforms into $\hat{p}_4(\tilde{x},\tilde{\epsilon})$ with the change of variables $x + \epsilon/2 = \tilde{x}$. Of these three functions, $\hat{p}_6(x,\epsilon)$ provides the best approximation for the plus function (see Figure 1, left).

In the following, we limit consideration to the three distinctive smooth plus functions $\hat{p}_1(x, \epsilon)$, $\hat{p}_5(x, \epsilon)$ and $\hat{p}_6(x, \epsilon)$. They are illustrated in Figure 1 (right) for different values of ϵ. We can see that, for the same value of ϵ, smooth plus function $\hat{p}_6(x, \epsilon)$ approximates the plus function with a better accuracy than the other two.

4 Numerical study

4.1 Performance of smooth plus functions in CVaR-minimising hedging

We compare the performance of three classes of smooth plus functions, discussed in Section 3, using an example of one period CVaR-minimising hedging. The performance and convergence of the smoothing method are compared against the solution obtained via linear programming [10, 11].

Consider a problem of hedging a short call option with the time to maturity of fifteen days and with the strike price of 100, using the underlying asset and another option on the underlying asset. The time period is $[0, t]$, and the price of the underlying asset is

$$s(t) = s_0 \exp\left[(\mu - \sigma^2/2)t + \sigma\sqrt{t}\Omega\right], \tag{13}$$

where s_0 is the initial price, μ and σ are the drift and the volatility of the underlying asset price, $\Omega \sim N(0, 1)$. The loss function associated with such a portfolio is

$$f(x, s) = \delta C - x_u \delta s - x_1 \delta C^{(1)}. \tag{14}$$

where x_u and x_1 are the numbers of units of the underlying asset and of the hedging option respectively, $C(s)$ and $C^{(1)}(s)$ are the prices of the target and the hedging option respectively, $\delta C = C(s(t)) - C(s_0)$, $\delta s = s(t) - s_0$, $\delta C^{(1)} = C^{(1)}(s(t)) - C^{(1)}(s_0)$. Note that, with this definition, a loss is positive,

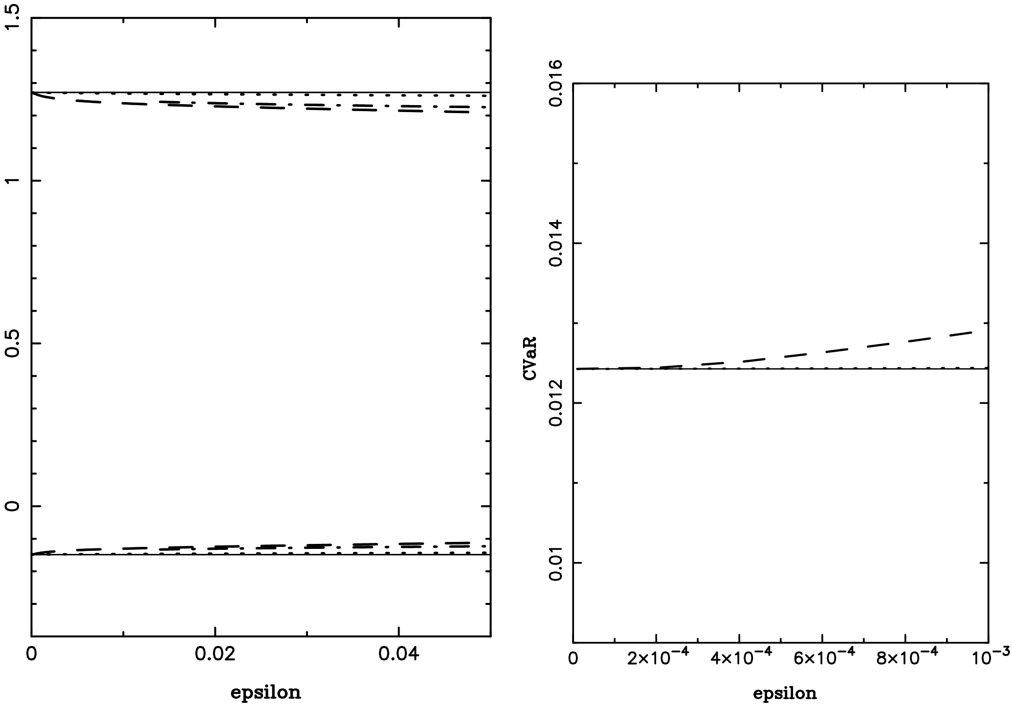

FIGURE 2: Convergence comparison for several smooth plus functions in CVaR minimisation. The left figure shows x_u (lower curves) and x_1 (upper curves) as functions of ϵ for $\hat{p}_6(x,\epsilon)$ (dotted curves), $\hat{p}_5(x,\epsilon)$ (dashed curves) and $\hat{p}_1(x,\epsilon)$ (dashed-dotted curves). LP solutions are shown with solid curves. The right figure shows CVaR as a function of ϵ for $\hat{p}_6(x,\epsilon)$ (dotted curve) and $\hat{p}_5(x,\epsilon)$ (dashed curve).

while a negative loss means a profit. The target option C is a short call with the strike $K = s_0$ and the time to maturity of ten days, while the hedging option $C^{(1)}$ is a call with the strike $K = s_0$ and the time to maturity of fifteen days. The time horizon is three days.

We generate a sample of M scenarios for the price of the underlying asset s^j, $j = 1, \dots, M$, via Monte Carlo simulations. The parameters of the model

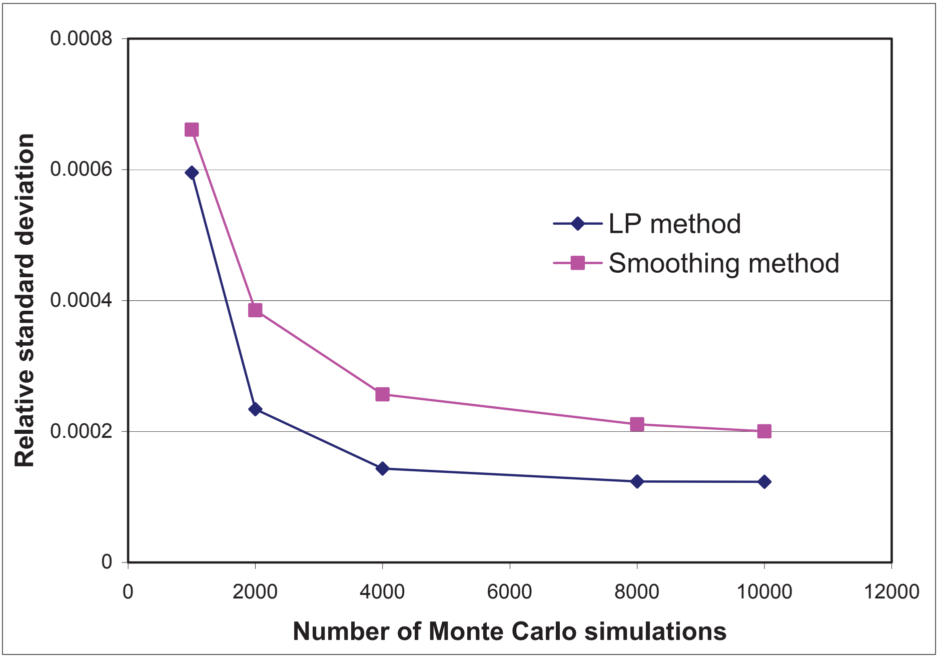

Figure 3: Relative standard deviations (rsd) of cVaR estimates. Blue curve shows the rsd of cVaR estimates from the LP method, while red curve shows the rsd of cVaR estimates from the smoothing method, with mean value of cVaR estimates from the LP method taken as a true estimate. The smoothing method used Zang smooth plus function (12) with $\epsilon = 10^{-3}$.

are taken as $\mu = 0.1$, $\sigma = 0.2$, $s_0 = 100$. The prices of call options are calculated according to the Black–Scholes formula with the risk-free interest rate $r = 0.04$. The probability level for cVaR used throughout the study is $\alpha = 0.95$. For the linear programming approach, we use the IMSL sparse LP solver SLPRS, whereas for smoothed minimisation we use the IMSL quasi-Newton optimiser BCONG.

Figure 2 shows the cVaR values (on the right) and the decision variables (on the left), calculated for various ϵ and for $M = 1000$. The cVaR and the decision variables from the smoothing method converge to LP solutions as $\epsilon \to 0$. All smooth plus functions showed robust performance in the quasi-Newton optimiser, with little dependence on the initial guess. However, the solution with $\hat{p}_3(x,\epsilon)$ could not be calculated for small values of ϵ ($\epsilon < 0.014$) because $e^{x/\epsilon}$ becomes very large (outside the machine precision).

One can see that the comparative convergence of the approximate optimal solutions for different smooth plus functions reflects the order in which these smooth plus functions approximate the plus function for a given ϵ (see Figure 1). Thus, the solution that uses the smooth plus function $\hat{p}_6(x,\epsilon)$ approximates the LP solution very closely for a wide range of values of ϵ, while the performance of $\hat{p}_1(x,\epsilon)$ and $\hat{p}_5(x,\epsilon)$ are not as good. In Figure 2, the approximation for CVaR with $\hat{p}_6(x,\epsilon)$ is virtually indistinguishable from the LP solution for $\epsilon < 10^{-3}$ (and higher values as well), while approximation for CVaR with $\hat{p}_5(x,\epsilon)$ is sufficiently close for $\epsilon < 2\times 10^{-4}$, but increases rapidly for larger values of ϵ. Optimisation with very small values of ϵ ($\epsilon < 10^{-7}$) becomes unreliable for all smooth plus functions.

The accuracy of the smoothing method depends on both the smoothing parameter ϵ and on the number of Monte Carlo simulations M. Therefore M and ϵ should be chosen in accordance with each other. We consider the CVaR estimate obtained via the LP method as a true estimate. As such an estimate itself incurs error due to the Monte Carlo sample size, the Monte Carlo error should be taken into account when selecting the appropriate value of ϵ.

We evaluate the estimation error as follows [17]. We compute the CVaR estimates using the LP method with sample sizes of 1000, 2000, 4000, 8000, and 10000. For each sample size, we generate 100 different samples and calculate the mean and the standard deviation of the CVaR estimate. We treat the mean value of CVaR as a true value and estimate the standard deviation of the smoothing method estimates using this mean value. We then compute the relative standard deviation (rsd) (the ratio of the standard deviation and the mean value) and use it to estimate the error of the smoothing method. Figure 3 shows the relative standard errors of the CVaR estimates as functions of Monte Carlo simulations number. The Zang smooth plus function (12) with the same value of the smoothing parameter $\epsilon = 10^{-3}$ was used in all calculations. Figure 3 shows that the accuracy of the smoothing method improves with increasing Monte Carlo sample size. The accuracy of the CVaR estimate from the LP method also improves with increasing Monte Carlo sample size. When selecting the suitable value of ϵ for a given M, ϵ should not be decreased beyond a particular level because otherwise the errors in the Monte Carlo simulations would dominate the calculations. The results shown in Figure 3 suggest that the same value of the smoothing parameter $\epsilon = 10^{-3}$ can be used for all Monte Carlo sample sizes in this example.

The reduction in computing time for the smoothed CVaR minimisation problem, as compared to LP, is dramatic. In the above example, the LP algorithm takes approximately 90 secs, whereas the smoothed optimisation method requires less than 0.1 secs.

4.2 Comparative study of one period CVaR-minimising and delta-gamma hedging

An efficient CVaR minimisation via the smoothing method makes the CVaR-minimising hedging a realistic alternative to the delta-gamma hedging commonly used in practice. We compare the performance of one period delta-gamma hedging and CVaR-minimising hedging. The model parameters are as in Section 4.1. We use $M = 1000$, as Figure 3 shows that this size of Monte Carlo sample provides sufficient accuracy in this example. The Zang smooth plus function (12) is used in smoothed CVaR minimisation.

In delta-gamma hedging, delta (Δ) measures the sensitivity of the value of an option to changes in the price of the underlying stock assuming all other parameters remain unchanged and is represented as the partial derivative of the option's fair value with respect to the price of the underlying security.

Table 1: Comparison of delta-gamma and CVaR-minimising hedges

Time horizon (days)	Hedge type	CVaR	Exp. return	Std deviation
1	delta-gamma	5.437E−3	−3.523E−3	1.273E−3
	min CVaR	4.339E−3	−3.659E−3	1.854E−3
2	delta-gamma	1.522E−2	−7.291E−3	5.389E−3
	min CVaR	1.093E−2	−7.851E−3	7.913E−3
3	delta-gamma	2.985E−2	−1.134E−2	1.283E−2
	min CVaR	2.057E−2	−1.263E−2	1.901E−2

When the change in the value of the underlying security is not small, the second order term cannot be ignored. Gamma (Γ) measures the rate of change in delta with respect to changes in the underlying price and is represented as the second derivative of the value with respect to the underlying price.

For the example (14) in Section 4.1, $\Delta = \partial\Pi/\partial s = x_1 \partial C^{(1)}/\partial s - \partial C/\partial s + x_u$, $\Gamma = \partial^2\Pi/\partial s^2 = -\partial^2 C/\partial s^2 + x_1 \partial^2 C^{(1)}/\partial s^2$. To eliminate instantaneous risk, positions need to be selected as

$$x_u = -x_1 \partial C^{(1)}/\partial s + \partial C/\partial s\,, \quad x_1 = \frac{\partial^2 C/\partial s^2}{\partial^2 C^{(1)}/\partial s^2}\,. \tag{15}$$

Numerical simulations show that a CVaR-minimising hedge always reduces the magnitude of large losses, compared to a delta-gamma hedge. However, there is often a trade-off between the CVaR and other performance criteria if the same volatility is used for the underlying asset and for the options. Table 1 gives examples where commonly used performance criteria, such as the CVaR, the expected return and the variance, are shown for delta-gamma and CVaR-minimising hedges for several time horizons. Table 1 shows that while the CVaR-minimising hedge delivers significant reduction in large losses, the delta-gamma hedge has larger expected return values and smaller variability (characterised by the standard deviation).

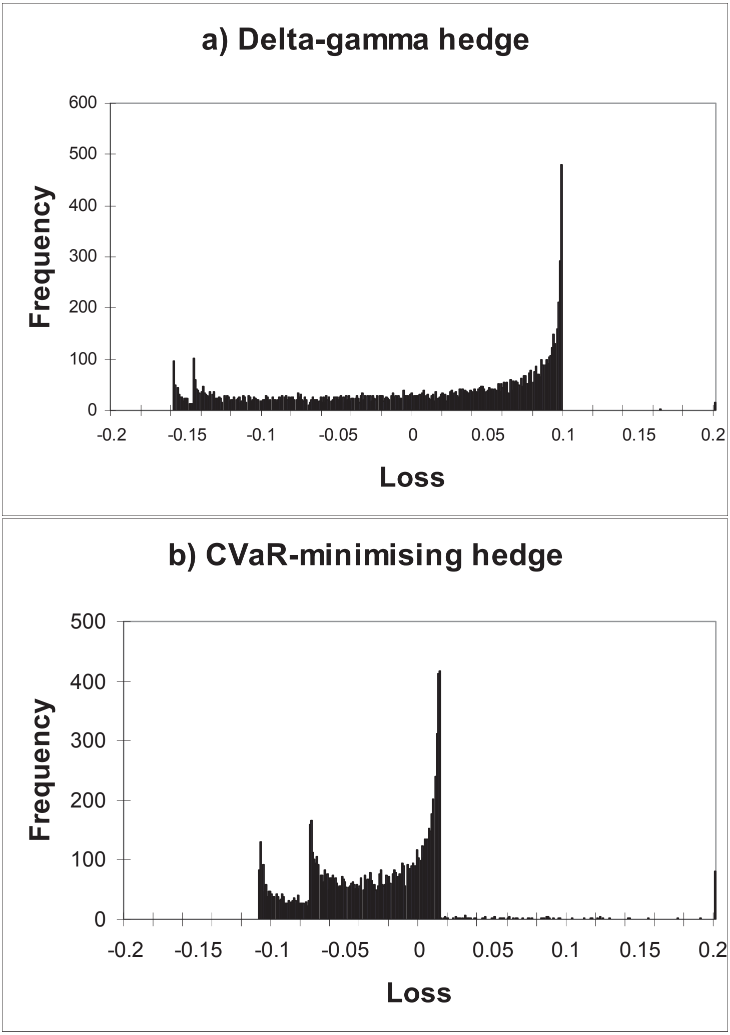

Figure 4: Loss distributions of the delta-gamma hedge and CVaR-minimising hedge; $\sigma = 0.5$, $\sigma_{\text{impl}} = 0.65$, hedging option with maturity of five days, strike 100, time horizon of three days, 1000 Monte Carlo simulations; a) CVaR is 9.91E−2, expected return is 7.77E−4, Standard deviation is 8.37E−2; b) CVaR is 4.51E−2, expected return is 3.19E−2, Standard deviation is 4.94E−2.

In practice, the implied volatility σ_{impl} for options is mostly different from the historical volatility σ of the underlying asset. Implied volatility is usually larger than the realised volatility of the underlying asset, because by using larger implied volatility when pricing options, traders can reduce the risk of making a loss in hedging. The performance of the CVaR-minimising hedge and the delta-gamma hedge with different volatilities for the underlying asset and for options on the underlying are compared in Figure 4 under the generally high volatility $\sigma = 0.5$, $\sigma_{\text{impl}} = 0.65$. Figure 4 shows that the loss distribution of the CVaR-minimising hedge has significantly shorter tail, larger expected return and smaller variability. Numerical simulations show that the advantage of the CVaR-minimising hedge increases with increase in both the historical volatility and the difference between the implied and the historical volatilities.

5 Conclusions

The performance of several smooth plus functions in one period CVaR minimising hedging is studied and it is shown that the Zang [18] smooth plus function (12) outperforms other smooth plus functions. The accuracy of the approximation improves with increasing Monte Carlo sample size and with decrease in the smoothing parameter. Due to the inherent error in Monte Carlo sampling, which grows as the sample size decreases, a decrease in the smoothing parameter value may not be required to improve the accuracy for smaller Monte Carlo samples.

By using the smoothing method, CVaR minimisation is sped up by a factor of 1000 in our test example, compared to the traditional LP method. This makes the CVaR-minimising hedging a realistic alternative to delta-gamma hedging in practice. Numerical simulations suggest that the CVaR-minimising hedging may outperform delta-gamma hedging according to several commonly used performance criteria, in the environment of a high volatility.

We considered the hedging of vanilla European options as an example, to demonstrate the effectiveness of the CVaR-minimising hedging approach. However, such an approach is not limited to European options and can be applied to various derivatives. While the Black–Scholes formula is used to price European options in this study, this does not mean that the underlying asset must follow a geometrical Brownian process (which is the assumption of the Black–Scholes–Merton methodology). In financial markets, as a convention, the Black–Scholes formula is used to express the price of vanilla European options through their quoted implied volatilities. The Black–Scholes formula is merely a short-hand to convert the quoted implied volatility into a price.

For the CVaR-minimising hedging, the underlying asset can assume any stochastic process, for example, a geometrical Brownian motion with jumps, and also a historical asset price movement can be readily assumed. CVaR-minimising hedging is not restricted to the assumption of Black–Scholes–Merton methodology, whereas the dynamic delta-gamma hedging implicitly relies on the same assumption as the pricing model (for example, the Black–Scholes–Merton assumption of a geometrical Brownian process for the underlying in the example above). We expect that the advantage of the CVaR-minimising hedging over the delta-gamma hedging may be more pronounced for jump-diffusion models, and for situations when historical data are used for the underlying price.

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Author addresses

1. **T. Tarnopolskaya**, CSIRO Mathematics, Informatics and Statistics, Sydney, Australia.
   mailto:tanya.tarnopolskaya@csiro.au
2. **Z. Zhu**, CSIRO Mathematics, Informatics and Statistics, Melbourne, Australia.