# Sparse inverse and characteristic polynomial of generalized arrow matrix

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## Abstract

A generalized arrow matrix of order $n$ with $m$ non-zero rows and columns is presented. If a simple condition holds, the inverse of this matrix is also an arrow matrix of the same form. We then derive a simple expression for its characteristic polynomial.

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# 1 Introduction

When testing numerical routines, a library of matrices with known inverses or other properties is useful. Most matrices with known inverses have none or at most two parameters that can be varied to provide a family of tests [3, 8]. More complex examples can be generated with the Sherman-Morrison formula or by Schur complements [2], but there appear to be very few simple test matrices of general size with many parameters. Notable exceptions to this are the Vandermonde and some tri-diagonal matrices [1, 3]. The arrow matrix [4, 6, 7] is another example; it is not hard to derive simple expressions for the inverse, determinant and characteristic polynomial of the arrow matrix which has the last row and column non-zero, and a non-zero diagonal [9].

We give a generalization of the arrow matrix, with an arbitrary number of non-zero columns and rows, whose inverse is also an arrow matrix, and we also give a simple expression for its characteristic polynomial. Such matrices and methods are also of interest as pre-conditioners for iterative processes, because of the sparsity of either the matrix or the inverse, and the freedom in choosing many of the elements.

# 2 Inverse of an arrow matrix

**Definition 1** *Let $M$ be the arrow matrix of order $n$*

$$M = \begin{bmatrix} D & e' \\ f & A \end{bmatrix} \tag{1}$$

*where $D$ is a diagonal matrix order $n-m$, $A$ is square of order $m$, $e$ and $f$ are $m \times (n-m)$ with constant rows, i.e.*

$$e = \begin{bmatrix} e_1 & e_1 & \cdots & e_1 \\ e_2 & e_2 & \cdots & e_2 \\ \vdots & \vdots & & \vdots \\ e_m & e_m & \cdots & e_m \end{bmatrix}, \qquad f = \begin{bmatrix} f_1 & f_1 & \cdots & f_1 \\ f_2 & f_2 & \cdots & f_2 \\ \vdots & \vdots & & \vdots \\ f_m & f_m & \cdots & f_m \end{bmatrix}.$$

In general the inverse of $M$ will be dense, however provided $D$ is diagonal and a simple condition is satisfied, $M^{-1}$ will be sparse, and indeed will be another arrow matrix. Let the inverse of $M$ be

$$M^{-1} = \begin{bmatrix} C & p' \\ q & B \end{bmatrix}.$$

Then using the usual formulae we obtain (see [5]) $C = (D - e'A^{-1}f)^{-1}$. Now for $M^{-1}$ to be an arrow matrix, we require that $C$ be diagonal. This implies that $D$ be diagonal and that $e'A^{-1}f = 0$.

**Theorem 1** *Assuming that $A^{-1}$ exists, the inverse of the arrow matrix $M$ is given by*

$$M^{-1} = \begin{bmatrix} D^{-1} & p' \\ q & B \end{bmatrix},$$

*where*

$$q = -A^{-1}fD^{-1},$$

$$p' = -D^{-1}e'A^{-1},$$

$$B = A^{-1} + qDp',$$

*provided*

$$e'A^{-1}f = 0. \tag{2}$$

*Writing the inverse as a matrix of cofactors, this condition can be transformed into*

$$\begin{vmatrix} 0 & \overline{e}' \\ \overline{f} & A \end{vmatrix} = 0,$$

*where $|D|$ denotes a determinant and by $\overline{f}$ we mean the first column of $f$, similarly for $\overline{e}$:*

$$\overline{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}, \qquad \overline{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}.$$

Matrices $p$ and $q$ have the same size and structure as $e$, that is constant rows. These matrices generate strikingly simple results, and have many pleasing properties, for example the orthogonality conditions hold:

$$e'(BA - I) = 0\,, \quad p'(B^{-1} - A) = 0\,.$$

**Example 1** *Here $|M| = -1$, $D$ unit diagonal:*

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 1 & 0 & 0 \\ -2 & -2 & -2 & -2 & -2 & 0 & 2 & 3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -8 & 7 & 3 \\ 0 & 1 & 0 & 0 & 0 & -8 & 7 & 3 \\ 0 & 0 & 1 & 0 & 0 & -8 & 7 & 3 \\ 0 & 0 & 0 & 1 & 0 & -8 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & -8 & 7 & 3 \\ -2 & -2 & -2 & -2 & -2 & 80 & -69 & -30 \\ 1 & 1 & 1 & 1 & 1 & -37 & 32 & 14 \\ 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 \end{bmatrix}. \tag{3}$$

Irrespective of the size $n$ of matrix $M$, if (2) is satisfied, its inverse will be an arrow matrix. The size of $D^{-1}$, $p$ and $q$ will depend on $n$ but their elements will not change, and only the elements of $B$ will depend on $n$.

# 3 Characteristic Polynomial

First, to find the determinant of $M$, we need a lemma. Note that in the remainder of this paper it is not necessary that (2) holds.

**Lemma 1** *Define the matrices* $Q_{n,m}$ *and* $Q$ *as*

$$Q_{n-m,m} = \begin{bmatrix} 0 & & & & e' \\ & d_2 & & & \vdots \\ & & \ddots & & \\ & & & d_{n-m} & \\ f & \dots & & & A \end{bmatrix}, \qquad Q = \left[\begin{array}{c|c} 0 & \overline{e}' \\ \hline f & A \end{array}\right],$$

*and put* $Q_{m+1,m} = Q$. *Then*

$$|Q_{n,m}| = \prod_{i=2}^{n-m} d_i \, |Q| \, , \quad n > m \, .$$

**Proof.** Expand $|Q_{m+i,m}|$ along the row $i$.

$$|Q_{m+i,m}| \;=\; d_i \begin{vmatrix} 0 & & & & e' \\ & d_2 & & & \vdots \\ & & \ddots & & \\ & & & d_{i-1} & \\ f & \dots & & & A \end{vmatrix} - e_1 \begin{vmatrix} 0 & & & & 0 & \tilde{e}' \\ 0 & d_2 & & & \vdots & \vdots \\ 0 & & \ddots & & 0 & \\ 0 & & & d_{i-1} & 0 & \\ f & \dots & & & & \tilde{A} \end{vmatrix} +$$

$$+ e_2 \begin{vmatrix} 0 & & & & 0 & \tilde{e}' \\ 0 & d_2 & & & \vdots & \vdots \\ 0 & & \ddots & & 0 & \\ 0 & & & d_{i-1} & 0 & \\ f & \dots & & & & \tilde{A} \end{vmatrix} + e_3 \cdots ,$$

where $\tilde{A}$ indicates one column has been deleted from a matrix $A$.

Because $f$ has constant rows, all the determinants are zero except the first, giving $|Q_{m+i,m}| = d_i \, |Q_{m+i-1,m}|$, and the lemma follows.

**Theorem 2** *The determinant of the arrow matrix (1) is given by*

$$|M| = |A| \prod_{j=1}^{n-m} d_j + |Q| \sum_{i=1}^{n-m} \prod_{j=1, j \neq i}^{n-m} d_j \, , \;\; n \geq m \, .$$

*If $D$ has a constant diagonal $d$, this simplifies to*

$$|M| = |A| d^{n-m} + |Q|(n-m) d^{n-m-1} \, ,$$

*with $Q$ as above.*

**Proof.** Expand determinant along top row:

$$|M| = d_1 \begin{vmatrix} d_2 & & & & e' \\ & d_3 & & & \vdots \\ & & \ddots & & \\ & & & d_{n-m} & \\ f & \dots & & & A \end{vmatrix} + \begin{vmatrix} 0 & & & & e' \\ & d_2 & & & \vdots \\ & & \ddots & & \\ & & & d_{n-m} & \\ f & \dots & & & A \end{vmatrix} .$$

Using the notation $F(n, m, d_1 : d_{n-m}) = |M|$, then

$$\begin{aligned} |M| &= F(n, m, d_1 : d_{n-m}) \\ &= d_1 \, F(n-1, m, d_2 : d_{n-m}) + Q_{n,m} \\ &= d_1 \, (d_2 \, F(n-2, m, d_3 : d_{n-m}) + Q_{n-1,m}) + Q_{n,m} \\ &= \prod_{j=1}^{n-m} d_j F(m, m, 0) + \prod_{j=1}^{n-m-1} d_j Q_{m+1,m} + \cdots + d_1 Q_{n-1,m} + Q_{n,m} \, . \end{aligned}$$

Applying the above lemma and the identity $F(m, m, 0) = |A|$ the result follows.

We can now write out the characteristic polynomial of $M$.

**Theorem 3** *The characteristic polynomial of the matrix $M$ defined as above is given by*

$$|A - \lambda I| \prod_{j=1}^{n-m} (d_j - \lambda) + |Q_\lambda| \sum_{i=1}^{n-m} \prod_{j=1, j \neq i}^{n-m} (d_j - \lambda) \, ,$$

*where*

$$Q_\lambda = \left[ \begin{array}{c|c} 0 & \vec{e}' \\ \hline \vec{f} & A - \lambda I \end{array} \right].$$

*If $D$ has a constant diagonal d, the characteristic polynomial is*

$$|A - \lambda I|\,(d-\lambda)^{n-m} + |Q_\lambda|\,(n-m)\,(d-\lambda)^{n-m-1}, \quad n \geq m\,.$$

For example, the eigenvalues of $M$ in Example 1 (for any $n$) are the roots of

$$(1-\lambda)^{n-4}\,\Big((1-\lambda)\,(-1-2\,\lambda+4\,\lambda^2-\lambda^3)+(n-3)\,(23\,\lambda-7\,\lambda^2)\Big)=0$$

with $n = 8$ in this case. Numerically the roots are $-5.09395$, $0.00879162$, 1, 1, 1, 1, $3.2824$, $6.80275$.

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