The Euler-Maclaurin formula revisited

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Abstract

The Euler-Maclaurin summation formula for the approximate evaluation of $I = \int_0^1 f(x)\,dx$ comprises a sum of the form $(1/m)\sum_{j=0}^{m-1} f((j+t_\nu)/m)$, where $0 < t_\nu \leq 1$, a second sum whose terms involve the difference between the derivatives of f at the end-points 0 and 1 and a truncation error term expressed as an integral. By introducing an appropriate change of variable of integration using a sigmoidal transformation of order $r > 1$, (other authors call it a periodizing transformation) it is possible to express I as a sum of m terms involving the new integrand with the second sum being

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zero. We show that for all functions in a certain weighted Sobolev space, the truncation error is of order $O\left(1/m^{n_1}\right)$, for some integer n_1 which depends on r. In principle we may choose n_1 to be arbitrarily large thereby giving a good rate of convergence to zero of the truncation error.

This analysis is then extended to Cauchy principal value and certain Hadamard finite-part integrals over $(0,1)$. In each case, the truncation error is $O\left(1/m^{n_1}\right)$. This result should prove particularly useful in the context of the approximate solution of integral equations although such discussion is beyond the scope of this paper.

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1 Introduction

The purpose of this paper is to review the Euler-Maclaurin formula and its application to the evaluation of ordinary, Cauchy principal value and certain Hadamard finite-part integrals over a finite interval which we shall take throughout to be $(0,1)$. Firstly, we shall, in §2, consider the Euler-Maclaurin formula for the so-called offset trapezoidal rule, particular cases of which give the well known trapezoidal and mid-point rules. Although Theorem 2.1 is well known we shall give the proof here and to that end we have gathered together at the end of this section some results on Bernoulli polynomials and periodic Bernoulli functions. In §3 we introduce the sigmoidal transformations which are so necessary for the approximate evaluation of these integrals and introduce a normed space of functions, denoted by K_α^N, in which we are able to do all our error analysis. Finally in §3 we consider the evaluation of ordinary integrals. §4 is concerned with Cauchy principal value integrals and in §5 we consider certain Hadamard finite-part integrals. In each case we show that the error, for an m point rule, converges to zero like $O\left(1/m^{n_1}\right)$ for some integer n_1 depending on the order of the sigmoidal transformation. This turns out to be a very satisfactory result.

The notation that we have adopted in this paper, owes a lot to that given by Lyness [8]; indeed, much of the work has been inspired by that paper. A full discussion on sigmoidal transformation is to be found in Elliott [3] and *asymptotic* estimates of the errors have been discussed in Elliott and Venturino [5].

Throughout this paper, **N** will denote the set of all natural numbers i.e. $\mathbf{N} = \{1, 2, 3, \ldots\}$, $\mathbf{N}_0$ will denote $\{0\} \cup \mathbf{N}$ and **Z** will denote the set of all integers, positive, negative and zero.

The Bernoulli numbers and Bernoulli polynomials play an important role in this analysis and because there are minor differences in the way these are defined by various authors we shall, for the sake of completeness, gather together some results here. For the record, we use the notation as given in Abramowitz and Stegun [1], Gradshteyn and Ryzhik [6], Olver [9] and Steffensen [10].

The Bernoulli polynomials $B_j(x)$ of degree j, $j \in \mathbf{N}_0$, are defined via a generating function as

$$\frac{te^{xt}}{e^t - 1} = \sum_{j=0}^{\infty} B_j(x) \frac{t^j}{j!}, \tag{1}$$

for $|t| < 2\pi$. In particular, we find $B_0(x) = 1$, $B_1(x) = x - 1/2$, $B_2(x) = x^2 - x + 1/6$, $B_3(x) = x^3 - 3x^2/2 + x/2$ etc. The Bernoulli numbers B_j are defined simply by $B_j = B_j(0)$ so that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$ etc. We find for all $j \in \mathbf{N}$ that $B_{2j+1} = 0$. Finally, we introduce the periodic Bernoulli functions $\bar{B}_j(x)$. These are defined by

$$\begin{aligned} \bar{B}_j(x) &:= B_j(x), \quad \text{for } 0 \le x < 1, \\ \bar{B}_j(x+1) &:= \bar{B}_j(x), \quad \text{for all } x \in \mathbf{R}; \end{aligned} \tag{2}$$

see Steffensen [10, §144]. In other words, $\bar{B}_j(x)$ takes the values of $B_j(x)$ on $[0,1)$ and is a periodic function with period 1. Except when $j = 1$, $\bar{B}_j(x)$ is continuous for all $x \in \mathbf{R}$. The function $\bar{B}_1(x)$ is a 'saw tooth' function with a finite jump discontinuity of magnitude 1 at each integer. We note that

$$\begin{aligned} \bar{B}_1(x) &= B_1(x), \quad \text{for } 0 \le x < 1 \quad \text{so that} \\ \bar{B}_1(1) &= B_1(0) = B_1 = -1/2\,. \end{aligned} \tag{3}$$

The Fourier expansions of the periodic Bernoulli functions are well known, see, for example [1, §23.1.16]. We have

$$\frac{\bar{B}_{2j-1}(x)}{(2j-1)!} = (-1)^j \sum_{k=1}^{\infty} \frac{2\sin(2k\pi x)}{(2\pi k)^{2j-1}} \tag{4}$$

for all $x \in \mathbf{R}$ when $j \ge 2$ and for $x \in \mathbf{R}\backslash\mathbf{Z}$ when $j = 1$. Also

$$\frac{\bar{B}_{2j}(x)}{(2j)!} = (-1)^{j-1} \sum_{k=1}^{\infty} \frac{2\cos(2k\pi x)}{(2\pi k)^{2j}} \tag{5}$$

for all $x \in \mathbf{R}$ and for all $j \in \mathbf{N}$. Finally, we note that

$$\frac{d}{dx}\bar{B}_{j+1}(x) = (j+1)\,\bar{B}_j(x) \tag{6}$$

for all $x \in \mathbf{R}$ when $j \ge 2$ and for all $x \in \mathbf{R}\backslash\mathbf{Z}$ when $j = 1$.

2 The off-set trapezoidal rule

Our aim initially is to find approximations to the integral If where

$$If := \int_0^1 f(x)\,dx\,. \tag{7}$$

Following Lyness [8], we define

$$t_\nu := (\nu + 1)/2 \quad \text{for } -1 < \nu \le 1\,. \tag{8}$$

The quadrature rule $Q_m^{[\nu]} f$ is defined by

$$Q_m^{[\nu]} f := \begin{cases} \dfrac{1}{m} \displaystyle\sum_{j=0}^{m-1} f\left((j + t_\nu)/m\right), & -1 < \nu < 1\,, \\ \dfrac{1}{m} \displaystyle\sum_{j=0}^{m}{}'' f(j/m), & \nu = 1\,. \end{cases} \tag{9}$$

Here $\sum''$ denotes a sum whose first and last terms are halved. We are interested in determining the error $If - Q_m^{[\nu]} f$ under various conditions on f. Lyness [8] quotes the following theorem, describing it as 'classical'.

Theorem 2.1 *Suppose f is such that for some $n \in \mathbf{N}$, $f^{(n-1)} \in C[0,1]$ and $f^{(n)} \in L_1(0,1)$. Then, for every $m \in \mathbf{N}$,*

$$\begin{aligned} If &= Q_m^{[\nu]} f - \sum_{j=1}^{n} \frac{\bar{B}_j(t_\nu)}{j!} \cdot \frac{f^{(j-1)}(1) - f^{(j-1)}(0)}{m^j} \\ &\quad + \frac{1}{m^n} \int_0^1 \frac{f^{(n)}(x)\,\bar{B}_n(t_\nu - mx)}{n!}\,dx\,. \end{aligned} \tag{10}$$

Proof. For $0 < t_\nu \le 1$, we define, on the interval $[0,1]$ the function P_1 by

$$P_1(x) := \begin{cases} x - t_\nu + 1/2\,, & 0 \le x < t_\nu\,, \\ x - t_\nu - 1/2\,, & t_\nu \le x \le 1\,. \end{cases} \tag{11}$$

This function has a finite jump discontinuity of magnitude 1 at the point t_ν. If $\mathbf{Z}$ denotes the set of all integers then we can extend the definition of P_1 to $\mathbf{R}\backslash(t_\nu + \mathbf{Z})$ by writing

$$\begin{aligned} \bar{P}_1(x) &= P_1(x)\,, & x \in [0,1]\backslash t_\nu\,, \\ \bar{P}_1(x+1) &= \bar{P}_1(x)\,, & x \in \mathbf{R}\backslash(t_\nu + \mathbf{Z})\,. \end{aligned} \tag{12}$$

Thus $\bar{P}_1$ is defined almost everywhere on $\mathbf{R}$ as a piecewise linear function of period 1 with a finite jump discontinuity of magnitude 1 at the points $t_\nu + \mathbf{Z}$. From (11) we find that the Fourier series expansion of P_1 is given by

$$P_1(x) = \sum_{k=1}^{\infty} \frac{2\sin(2\pi k(t_\nu - x))}{2\pi k}\,, \quad \text{for all } x \in [0,1]\backslash t_\nu\,. \tag{13}$$

We can use the Fourier series expansion to define $\bar{P}_1$ on $\mathbf{R}\backslash(t_\nu + \mathbf{Z})$ and, from (4), we see that

$$\bar{P}_1(x) = \sum_{k=1}^{\infty} \frac{2\sin(2\pi k(t_\nu - x))}{2\pi k} = -\bar{B}_1(t_\nu - x)\,, \tag{14}$$

for all $x \in \mathbf{R}\backslash(t_\nu + \mathbf{Z})$. As we shall see below, we shall require functions $\bar{P}_s$ for all $s \in \mathbf{N}$ such that when $s \ge 2$, $\bar{P}'_{s+1}(x) = \bar{P}_s(x)$ for all $x \in \mathbf{R}$. When

$s = 1$, we require this to be satisfied for all $x \in \mathbf{R}\backslash\left(t_\nu + \mathbf{Z}\right)$. From the well known properties of the periodic Bernoulli functions we see that we can do this by choosing

$$\bar{P}_s\left(x\right) = \left(-1\right)^s \bar{B}_s\left(t_\nu - x\right)/s!\,, \quad s \in \mathbf{N}\,. \tag{15}$$

With these preliminaries established let us consider the proof of (10). On choosing any $m \in \mathbf{N}$, from the definition of $\bar{P}_1$ we have

$$\bar{P}_1\left(mx\right) = \begin{cases} mx - \left(k + t_\nu\right) + 1/2\,, & \text{for } k/m \le x < \left(k + t_\nu\right)/m\,, \\ mx - \left(k + t_\nu\right) - 1/2\,, & \text{for } \left(k + t_\nu\right)/m \le x \le \left(k + 1\right)/m\,, \end{cases} \tag{16}$$

for all $k \in \mathbf{Z}$. On integrating the function $f'\left(x\right)\bar{P}_1\left(mx\right)$ by parts over the interval $\left[k/m,\, \left(k+1\right)/m\right]$, we find using (15) and (16) that

$$\begin{aligned} \int_{k/m}^{(k+1)/m} f\left(x\right)dx &= \frac{1}{m} f\left(\frac{k + t_\nu}{m}\right) - \frac{\bar{B}_1\left(t_v\right)}{m}\left[f\left(\frac{k+1}{m}\right) - f\left(\frac{k}{m}\right)\right] \\ &\quad - \frac{1}{m}\int_{k/m}^{(k+1)/m} \bar{P}_1\left(mx\right) f'\left(x\right)dx\,. \end{aligned} \tag{17}$$

On summing from $k = 0$ to $\left(m - 1\right)$ we find that

$$\begin{aligned} \int_0^1 f\left(x\right)dx &= \frac{1}{m}\sum_{k=0}^{m-1} f\left(\frac{k + t_\nu}{m}\right) - \frac{\bar{B}_1\left(t_\nu\right)}{m}\left[f\left(1\right) - f\left(0\right)\right] \\ &\quad - \frac{1}{m}\int_0^1 \bar{P}_1\left(mx\right) f'\left(x\right)dx\,. \end{aligned} \tag{18}$$

If $n \geq 2$, we can again integrate by parts the integral on the right hand side of (18). Recalling that $\bar{P}_2'(x) = \bar{P}_1(x)$ for all $x \in \mathbf{R} \setminus (t_\nu + \mathbf{Z})$ we find that

$$\int_0^1 \bar{P}_1(mx)\, f'(x)\, dx = \frac{\bar{B}_2(t_\nu)}{2!} \cdot \frac{f'(1) - f'(0)}{m} - \frac{1}{m} \int_0^1 \bar{P}_2(mx)\, f''(x)\, dx\,, \tag{19}$$

since $\bar{P}_2(m) = \bar{P}_2(0)$ by periodicity and $\bar{P}_2(0) = B_2(t_\nu)/2!$ by (15). Repeating this process an appropriate number of times and using (15), we recover (10) and the theorem is proved. ♠

Corollary 2.2 *Under the conditions of Theorem 2.1,*

$$\begin{aligned} If &= Q_m^{[\nu]} f - \sum_{j=1}^{n-1} \frac{\bar{B}_j(t_\nu)}{j!} \cdot \frac{f^{(j-1)}(1) - f^{(j-1)}(0)}{m^j} \\ &\quad - \frac{1}{m^n} \int_0^1 f^{(n)}(x) \left(\frac{\bar{B}_n(t_\nu) - \bar{B}_n(t_\nu - mx)}{n!} \right) dx\,. \end{aligned} \tag{20}$$

Proof. This follows immediately from (10) since

$$\begin{aligned} \frac{1}{m^n} \int_0^1 \frac{f^{(n)}(x)\, \bar{B}_n(t_\nu - mx)}{n!} dx &= \frac{1}{m^n} \int_0^1 \frac{\bar{B}_n(t_\nu - mx) - \bar{B}_n(t_\nu)}{n!} f^{(n)}(x)\, dx \\ &\quad + \frac{\bar{B}_n(t_\nu)}{m^n n!} \left(f^{(n-1)}(1) - f^{(n-1)}(0) \right)\,. \end{aligned} \tag{21}$$

Substituting (21) into (10) gives (20). ♠

Note that equation (20) is often quoted in the context of numerical analysis since one can readily apply the mean value theorem to the integral on the right hand side.

There are two choices of t_ν which are worthy of special note. When $t_\nu = 1$, although $\bar{B}_1(1) = -1/2$, we have $\bar{B}_{2s+1}(1) = \bar{B}_{2s+1}(0) = 0$ for all $s \in \mathbf{N}$. Again, when $t_\nu = 1/2$, we have that $B_{2s-1}(1/2) = 0$ for all $s \in \mathbf{N}$. These observations give rise to the following corollary.

Corollary 2.3 *Suppose that* $f^{(2n-1)} \in C[0,1]$ *and* $f^{(2n)} \in L_1(0,1)$ *for some* $n \in \mathbf{N}$. *Then, for every* $m \in \mathbf{N}$,

(a) when $\nu = 1$, *so that* $t_\nu = 1$,

$$\begin{aligned} If &= \frac{1}{m}\sum_{j=0}^{m}{}'' f(j/m) - \sum_{j=1}^{n}\frac{B_{2j}}{(2j)!}\cdot\frac{f^{(2j-1)}(1) - f^{(2j-1)}(0)}{m^{2j}} \\ &\quad + \frac{1}{m^{2n}}\int_0^1 \frac{f^{(2n)}(x)\,\bar{B}_{2n}(mx)}{(2n)!}dx\,; \end{aligned} \tag{22}$$

(b) when $\nu = 0$, *so that* $t_\nu = 1/2$,

$$\begin{aligned} If &= \frac{1}{m}\sum_{j=0}^{m-1} f((j+1/2)/m) - \sum_{j=1}^{n}\frac{B_{2j}(1/2)}{(2j)!}\cdot\frac{f^{(2j-1)}(1) - f^{(2j-1)}(0)}{m^{2j}} \\ &\quad + \frac{1}{m^{2n}}\int_0^1 \frac{f^{(2n)}(x)\,\bar{B}_{2n}(1/2-mx)}{(2n)!}dx\,. \end{aligned} \tag{23}$$

Proof. This follows immediately by substitution into equation (10). ♠

Note that, equally well, we could have put $t_\nu = 1/2$ and 1 into equation (20). The first summation on the right hand side of equation (22) is known as the 'trapezoidal rule'; that for equation (23) is known as the 'mid-point' rule.

All the above results are well known. We note from equation (10), for example, that if $f^{(j-1)}(1) - f^{(j-1)}(0) = 0$ for $j = 1\,(1)\,n$ then the error $If - Q_m^{[\nu]}f$ is of order $O\left(1/m^n\right)$. The larger we can choose n, the faster will be the rate of convergence to zero of the error. Thus if we have some degree of periodicity of the function f we could have a good rate of convergence to zero of the error. We shall exploit this idea in the subsequent sections. To do this we shall introduce, in the next section, a suitable space of functions and also the so-called 'sigmoidal transformations' which can impose some degree of periodicity on to our original (non-periodic) function.

3 A space of functions and sigmoidal transformations

To consider the error term in Theorem 2.1, it is convenient to introduce a space of functions similar to that previously considered by Kress [7] and Elliott and Prössdorf [4]. Essentially we consider functions which are 'smooth enough' on the open interval $(0, 1)$ and have singularities only at the end points 0 and 1. In

Definition 3.1 we shall let $\overset{\circ}{C}\,[0,1]$ denote the space of functions which are continuous on the compact interval $[0,1]$ and vanish at both end-points.

Definition 3.1 *Suppose α is a positive non-integer such that $n<\alpha<n+1$ for some $n\in\mathbf{N}_0$. Assume N is such that $\mathbf{N}\ni N\gg\alpha$. A function f is said to be in the space K_α^N if*

(i) $f\in C^{(N)}(0,1)$;

(ii) $f^{(j)}\in\overset{\circ}{C}\,[0,1]$, *for* $j=0\,(1)\,(n-1)$;

(iii) $\int_0^1 (t\,(1-t))^{j-\alpha}\left|f^{(j)}(t)\right| dt<\infty$, *for* $j=0\,(1)\,N$.

In addition, a norm on K_α^N will be denoted and defined by

$$\|f\|_{\alpha,N} := \max_{j=0(1)N}\int_0^1 (t\,(1-t))^{j-\alpha}\left|f^{(j)}(t)\right| dt\,. \tag{24}$$

Comments on the definition. It simplifies, a little, some of the subsequent results and does not affect the generality of the analysis, if we assume that α is not an integer. The choice of N 'much larger' than α is included so that f will be 'sufficiently smooth' on the open interval $(0,1)$ for all the subsequent analysis. The context will always make it clear how large N needs to be. Suppose we define a function $\hat{f}$ say, on $\mathbf{R}$ such that it takes the value of f on the open interval $(0,1)$ and satisfies $\hat{f}(1+x) = \hat{f}(x)$. As a consequence of (ii), we see that if $f \in K_\alpha^N$ then $\hat{f} \in C^{(n-1)}(\mathbf{R})$ so that $\hat{f}$ and all its derivatives up to order $(n-1)$ will be continuous and periodic (of period 1) on $\mathbf{R}$. This will be important in the context of Theorem 2.1. Again, since the Euler-Maclaurin formula (equation (10)) involves $f^{(j-1)}(1) - f^{(j-1)}(0)$ for $j = 1\,(1)\,n$, we treat the two end-points in the same way. That is, there appears to be no point in replacing (iii) in Definition 3.1 by, for example,

$$\int_0^1 t^{j-\alpha}(1-t)^{j-\beta} \left| f^{(j)}(t) \right| dt < \infty \,, \quad \text{for } j = 0\,(1)\,N \,,$$

for some β different from α.

As an immediate consequence of Definition 3.1, we have the following theorem.

Theorem 3.2 *Suppose $f \in K_\alpha^N$ where $\alpha \in (n, n+1)$ and $n \in \mathbf{N}_0$. Then*

(i) $f^{(j)}$ is integrable on $(0,1)$ for $j = 0\,(1)\,n$, and $\int_0^1 \left| f^{(j)}(t) \right| dt \le c \left\| f \right\|_{\alpha,N}$ for some positive constant c;

(ii) there exists a positive constant c such that

$$\left| f^{(j)}(t) \right| \le c\,(t\,(1-t))^{\alpha-(j+1)} \quad \textit{for } \; j = 0\,(1)\,(N-1) \,. \tag{25}$$

Note: Throughout the paper c will be used to denote a generic constant whose value may change from time to time. The context should make it clear what parameters c does not depend upon. Thus in (25) c will be independent of j and t but will depend on α.

Proof.

(i) We have, for $j = 0\,(1)\,n$,

$$\begin{aligned}
\int_0^1 \left| f^{(j)}(t) \right| dt &= \int_0^1 (t\,(1-t))^{j-\alpha} \left| f^{(j)}(t) \right| \cdot (t\,(1-t))^{\alpha-j}\, dt \\
&\le c \int_0^1 (t\,(1-t))^{j-\alpha} \left| f^{(j)}(t) \right| dt\,, \quad \text{since } \alpha - j > 0\,, \\
&\le c\,\|f\|_{\alpha,N}\,, \quad \text{by (24)}, \\
&< \infty\,.
\end{aligned}$$

(ii) Suppose first that $j = 0\,(1)\,(n-1)$. From (ii) of Definition 3.1 we have

$$\begin{aligned}
f^{(j)}(t) &= \int_0^t f^{(j+1)}(s)\, ds\,, \text{ since } f^{(j)}(0) = 0\,, \\
&= \int_0^t (s\,(1-s))^{j+1-\alpha} f^{(j+1)}(s) \cdot (s\,(1-s))^{\alpha-(j+1)}\, ds\,.
\end{aligned}$$

Now on $[0, 1/2]$, since $j = 0\,(1)\,(n-1)$ and $n < \alpha < n+1$, the function $(s\,(1-s))^{\alpha-(j+1)}$ is monotonic increasing so that

$$\left| f^{(j)}(t) \right| \le (t\,(1-t))^{\alpha-(j+1)} \int_0^t (s\,(1-s))^{j+1-\alpha} \left| f^{(j+1)}(s) \right| ds$$

$$\leq \quad (t\,(1-t))^{\alpha-(j+1)}\,\|f\|_{\alpha,N}\ , \qquad \text{on using (24).}$$

Again, on $[1/2, 1]$ we have, since $f^{(j)}(1) = 0$,

$$\left|f^{(j)}(t)\right| \leq \int_t^1 (s\,(1-s))^{j+1-\alpha}\left|f^{(j+1)}(s)\right| \cdot (s\,(1-s))^{\alpha-(j+1)}\,ds\,.$$

Since $(s\,(1-s))^{\alpha-(j+1)}$ is monotonic decreasing on $[1/2, 1]$ we can again argue as above and establish (25) on $[0, 1]$ for $j = 0\,(1)\,(n-1)$.

Suppose now that $j = n\,(1)\,(N-1)$. From

$$f^{(j)}(t) = f^{(j)}(1/2) + \int_{1/2}^t f^{(j+1)}(s)\,ds$$

we have for $t \in [0, 1/2]$ that

$$\left|f^{(j)}(t)\right| \leq \left|f^{(j)}(1/2)\right| + \int_t^{1/2}\left|f^{(j+1)}(s)\right|ds\,. \tag{26}$$

Since $(t\,(1-t))^{j+1-\alpha}$ is monotonic increasing on $[0, 1/2]$ and bounded above by $2^{-2(j+1-\alpha)}$ on that interval we have

$$\begin{aligned}(t\,(1-t))^{j+1-\alpha}\left|f^{(j)}(t)\right| \quad &\leq \quad 2^{-2(j+1-\alpha)}\left|f^{(j)}(1/2)\right| \\ &\qquad + \int_t^{1/2}(s\,(1-s))^{j+1-\alpha}\left|f^{(j+1)}(s)\right|ds \\ &\leq \quad 2^{-2(j+1-\alpha)}\left|f^{(j)}(1/2)\right| + \|f\|_{\alpha,N}\ .\end{aligned}$$

Consequently on $[0,1/2]$ we can find a positive constant c, independent of t, such that,

$$\left|f^{(j)}(t)\right| \leq c\,(t\,(1-t))^{\alpha-(j+1)}\ .$$

One may argue in a similar fashion on $[1/2,1]$ but the details will not be given. This establishes the theorem. ♠

It is appropriate here to consider how the Euler-Maclaurin formula applies to a function in K_α^N.

Theorem 3.3 *Suppose* $f \in K_\alpha^N$ *where* $n < \alpha < n+1$ *for some* $n \in \mathbf{N}$. *Then, for all* $m \in \mathbf{N}$,

$$If = Q_m^{[\nu]}f + E_m^{[\nu]}f \tag{27}$$

where

$$\left|E_m^{[\nu]}f\right| \leq \frac{c}{m^n}\,\|f\|_{\alpha,N} \tag{28}$$

for some positive constant c *independent of* m.

Proof. From the assumption that $f \in K_\alpha^N$ we have immediately that $f^{(j)}(0) = f^{(j)}(1) = 0$ for $j = 0\,(1)\,(n-1)$. Certainly $f^{(n-1)} \in C\,[0,1]$ and by Theorem 3.2(i) we have $f^{(n)} \in L_1\,(0,1)$. Since the conditions of Theorem 2.1 are satisfied we have

$$If = Q_m^{[\nu]}f + \frac{1}{m^n}\int_0^1 \frac{f^{(n)}(x)\,\bar{B}_n\,(t_\nu - mx)}{n!}dx\,.$$

But from (4) and (5) we have

$$\left|\bar{B}_n\left(t_\nu - mx\right)/n!\right| \leq 2\zeta\left(n\right)/\left(2\pi\right)^n ,$$

where ζ denotes the Riemann zeta function, so that

$$\left|E_m^{[\nu]}f\right| \leq \frac{c}{m^n}\int_0^1 \left|f^{(n)}\left(t\right)\right| dt \leq \frac{c}{m^n}\left\|f\right\|_{\alpha,N} ,$$

from Theorem 3.2(i), and the theorem is established. ♠

The question now arises as to how, given a function f, we may transform it so that a reasonable number of derivatives of the transformed function vanish at the end-points. We introduce a sigmoidal transformation γ_r, say, of order $r > 1$, which is a one-to-one mapping of the compact interval $[0, 1]$ onto itself. Following Elliott [3], we introduce the following Definition.

Definition 3.4 *A real-valued function γ_r is said to be a* sigmoidal transformation of order $r \geq 1$ *if the following conditions are satisfied:*

(i) $\gamma_r \in C^1\left[0,1\right] \cap C^\infty\left(0,1\right)$ *with* $\gamma_r\left(0\right) = 0$*;*

(ii) $\gamma_r\left(x\right) + \gamma_r\left(1-x\right) = 1, 0 \leq x \leq 1$*;*

(iii) γ_r *is strictly increasing on* $[0, 1]$*;*

(iv) γ_r' *is strictly increasing on* $[0, 1/2]$ *with* $\gamma_r'\left(0\right) = 0$*;*

(v) $\gamma_r^{(j)}(x) = O\left(x^{r-j}\right)$ *near* $x = 0$, $j \in \mathbf{N}_0$.

Let us recall that our aim is to find approximate values of the integral $If = \int_0^1 f(x)\,dx$. If we make the change of variable $x = \gamma_r(t)$ then we have immediately that

$$If = \int_0^1 g_r(t)\,dt \quad \text{say, where } g_r(t) = f(\gamma_r(t))\,\gamma_r'(t)\ . \tag{29}$$

It is to the function g_r that we shall apply Theorem 2.1 and in particular we shall suppose that

$$If = Q_m^{[\nu,r]}f + E_m^{[\nu,r]}f \tag{30}$$

say, where

$$Q_m^{[\nu,r]}f := \begin{cases} \dfrac{1}{m}\displaystyle\sum_{j=0}^{m-1} \gamma_r'\left((j+t_\nu)/m\right) f\left(\gamma_r\left((j+t_\nu)/m\right)\right)\ , & -1 < \nu < 1\,, \\ \dfrac{1}{m}\displaystyle\sum_{j=0}^{m}{}'' \gamma_r'(j/m)\, f\left(\gamma_r(j/m)\right)\ , & \nu = 1\,. \end{cases} \tag{31}$$

We note that for $-1 < \nu \le 1$

$$Q_m^{[\nu,r]}f = Q_m^{[\nu]}g_r\,, \quad \text{and} \quad E_m^{[\nu,r]}f := E_m^{[\nu]}g_r\,. \tag{32}$$

Our aim is to choose r such that $g_r \in K_\beta^N$ say where β is large enough so that we may apply Theorem 3.3 to g_r and obtain a good rate of convergence of $E_m^{[\nu,r]}f$ to zero as $m \to \infty$. Let us now be more precise.

Theorem 3.5 *Suppose $f \in K_\alpha^N$, for some non-integer $\alpha > 0$ and let γ_r be a sigmoidal transformation of order $r \geq 1$. Let*

$$g_r\left(t\right) := f\left(\gamma_r\left(t\right)\right)\gamma_r'\left(t\right) \ , \tag{33}$$

and suppose

$$\beta := \alpha r \quad \textit{with} \quad \beta \notin \mathbf{N}\, . \tag{34}$$

If we assume $\beta < N$ then

(i) $g_r \in K_\beta^N$,

(ii) there exists a positive constant c such that

$$\|g_r\|_{\beta,N} \leq c\,\|f\|_{\alpha,N} \ . \tag{35}$$

Proof. Let us suppose that $n_1 < \beta < n_1 + 1$ for some $n_1 \in \mathbf{N}$. Firstly we shall show that $g_r^{(j)}(0) = g_r^{(j)}(1) = 0$ for $j = 0\,(1)\,(n_1 - 1)$. By Leibnitz' theorem applied to the definition of g_r as given in (33) we have

$$g_r^{(j)}\left(t\right) = \sum_{s=0}^{j} \binom{j}{s} \left(\gamma_r'\left(t\right)\right)^{(j-s)} \left(f\left(\gamma_r\left(t\right)\right)\right)^{(s)} \ .$$

Let us rewrite this as

$$g_r^{(j)}\left(t\right) = \sum_{s=0}^{j} u_{s,j}\left(t\right) f^{(s)}\left(\gamma_r\left(t\right)\right) \ , \tag{36}$$

say. On differentiating (36) again with respect to t we find that the functions $u_{s,j}$ must satisfy

$$\begin{cases} u_{0,j+1}(t) = u'_{0,j}(t) \ , \\ u_{s,j+1}(t) = u_{s-1,j}(t)\,\gamma'_r(t) + u'_{s,j}(t) \ , \quad s = 1\,(1)\,j \ , \\ u_{j+1,j+1}(t) = \gamma'_r(t)\,u_{j,j}(t) \ . \end{cases} \tag{37}$$

In particular, we see that

$$u_{0,j}(t) = \gamma^{(j+1)}(t) \ , \quad u_{j,j}(t) = \left(\gamma'_r(t)\right)^{j+1} \ . \tag{38}$$

Let us consider what happens in a neighbourhood of $t = 0$. We have $u_{0,j}(t) = O\left(t^{r-j-1}\right)$ and $u_{j,j}(t) = O\left(t^{(r-1)(j+1)}\right)$. If we conjecture that, near $t = 0$,

$$u_{s,j}(t) = O\left(t^{r-1+rs-j}\right) \quad \text{for } \ s = 0(1)j \tag{39}$$

we see that this satisfies the special cases $s = 0$ and $s = j$ as well as the recurrence relations (37). Consequently we shall take the behaviour of $u_{s,j}$ near $t = 0$ to be given by (39). If we recall (25), we see that near $t = 0$ we have

$$f^{(j)}(t) = O\left(t^{\alpha-j-1}\right) \quad \text{for } \ j = 0\,(1)\,(N-1) \ . \tag{40}$$

Since $\gamma_r(t) = O(t^r)$ near $t = 0$ we have, from (36), (39) and (40), that

$$g_r^{(j)}(t) = O\left(t^{\alpha r-j-1}\right) \quad \text{for } \ j = 0\,(1)\,(N-1) \ . \tag{41}$$

In particular we have $g_r^{(j)}(0) = 0$ for all j such that $j < \alpha r - 1$. Recalling that $n_1 < \alpha r < n_1 + 1$, we have that $g_r^{(j)}(0) = 0$ for $j = 0\,(1)\,(n_1 - 1)$. We may argue similarly at the end point $t = 1$ so that we have $g^{(j)} \in \overset{\circ}{C} [0,1]$ for $j = 0\,(1)\,(n_1 - 1)$. This shows that (ii) of Definition 3.1 is satisfied; we must now show that

$$\int_0^1 (t\,(1-t))^{j-\beta} \left| g_r^{(j)}\,(t) \right| dt < \infty\,, \quad \text{for} \quad j = 0\,(1)\,N\,,$$

and in so doing we shall prove both (i) and (ii).

From (36),

$$\int_0^1 (t\,(1-t))^{j-\beta} \left| g_r^{(j)}\,(t) \right| dt \leq \sum_{s=0}^{j} \int_0^1 (t\,(1-t))^{j-\beta} \left| u_{s,j}\,(t) \right| \left| f^{(s)}\,(\gamma_r\,(t)) \right| dt\,. \tag{42}$$

Firstly, from (39), and a similar result valid near $t = 1$, we can write

$$u_{s,j}\,(t) = (t\,(1-t))^{r-1+rs-j}\, U_{s,j}\,(t) \tag{43}$$

say, where $U_{s,j}$ is continuous on $[0,1]$ and does not vanish at the end-points. In a similar vein we write

$$\gamma_r'\,(t) = (t\,(1-t))^{r-1}\, \Gamma_{r,1}\,(t) \tag{44}$$

and

$$\gamma_r\,(t)\,(1-\gamma_r\,(t)) = (t\,(1-t))^{r}\, \Gamma_{r,0}\,(t) \tag{45}$$

say, where the functions $\Gamma_{r,0}$ and $\Gamma_{r,1}$ are continuous and strictly positive on $[0,1]$. From (42)–(45) we find

$$\int_0^1 (t\,(1-t))^{j-\beta} \left| g_r^{(j)}\,(t) \right| dt$$

$$
\begin{aligned}
&\leq c \sum_{s=0}^{j} \int_0^1 (t(1-t))^{r-1+rs-\beta} \left| f^{(s)}(\gamma_r(t)) \right| dt \\
&\leq c \sum_{s=0}^{j} \int_0^1 \gamma_r'(t) (\gamma_r(t)(1-\gamma_r(t)))^{s-\beta/r} \left| f^{(s)}(\gamma_r(t)) \right| dt \\
&\leq c \sum_{s=0}^{j} \int_0^1 (x(1-x))^{s-\alpha} \left| f^{(s)}(x) \right| dx \,, \quad \text{on writing } \gamma_r(t) = x \,, \\
&\leq c(j+1) \|f\|_{\alpha,N} < \infty \,, \quad \text{for} \quad j = 0(1)N \,. \qquad (46)
\end{aligned}
$$

Hence condition (iii) of Definition 3.1 is satisfied so that $g_r \in K_\beta^N$ and furthermore we see from (46) that there exists a constant c such that $\|g_r\|_{\beta,N} \leq c\|f\|_{\alpha,N}$, and the theorem is proved. ♠

We are now in a position to put an upper bound on the error term $E_m^{[\nu,r]} f$, see (32).

Theorem 3.6 *Suppose $f \in K_\alpha^N$ for some non-integer $\alpha > 0$. Let γ_r be a sigmoidal transformation of order $r \geq 1$ such that $n_1 < \alpha r < n_1 + 1$, for some $n_1 \in \mathbf{N}$. Then with*

$$
I f = Q_m^{[\nu,r]} f + E_m^{[\nu,r]} f \,,
$$

there exists a positive constant c independent of m such that

$$
\left| E_m^{[\nu,r]} f \right| \leq \frac{c}{m^{n_1}} \|f\|_{\alpha,N} \,. \qquad (47)
$$

Proof. From (32), Theorem 3.3 and Theorem 3.5 we have

$$\left|E_m^{[\nu,r]}f\right| = \left|E_m^{[\nu]}g_r\right| \leq \frac{c}{m^{n_1}}\left\|g_r\right\|_{\beta,N} \leq \frac{c}{m^{n_1}}\left\|f\right\|_{\alpha,N} . \tag{48}$$

♠

This is the principal result of this section from which we see that with the introduction of a sigmoidal transformation, the application of the trapezoidal rule to the transformed integrand gives a higher rate of convergence than without the transformation.

Although we have assumed so far in this section that f is sufficiently smooth on $(0,1)$ (i.e. $f^{(N)} \in C(0,1)$) nevertheless we will now show that the quadrature rule as defined by equations (30) and (31) is convergent when f is merely continuous on $[0,1]$.

Theorem 3.7 *Suppose f is continuous on $[0,1]$. Then, with any sigmoidal transformation of order r where $r > 1$, $\lim_{m\to\infty} Q_m^{[\nu,r]}f = If$. i.e. the quadrature sum converges to the integral.*

Proof. This follows from the Pólya-Steckloff theorem see, for example Davis [2, p.353], which states that if a quadrature rule $Q_m f := \sum_{j=0}^{m-1} a_{j,m} f\left(x_{k,m}\right)$ has $a_{j,m} > 0$ for $j = 0(1)(m-1)$, $m \in \mathbf{N}$, then $\lim_{m\to\infty} Q_m f = \int_0^1 f(x)\,dx$ for all $f \in C[0,1]$ iff $\lim_{m\to\infty} Q_m(x^k) = \int_0^1 x^k dx$ for all $k \in \mathbf{N}_0$.

From the quadrature rule $Q_m^{[\nu,r]}f$ defined in (31) we can see that

$$a_{j,m} = (1/m)\,\gamma_r'\left(\left(j+t_\nu\right)/m\right)\ ,\quad j = 0\,(1)\,(m-1)\ ,$$

which is strictly positive, see Definition 3.4. It remains to consider the functions x^k for all $k \in \mathbf{N}_0$. These functions are in K_α^N for any $\alpha \in (0,1)$. Thus if $r > 1$ we can choose an $\alpha \in (1/r, 1)$ such that $1 < \alpha r < 2$ and consequently $n_1 = 1$. By Theorem 3.6

$$\left|E_m^{[\nu,r]}x^k\right| \le (c/m)\left\|x^k\right\|_{\alpha r,N}$$

for all $k \in \mathbf{N}_0$. Hence $\lim\limits_{m\to\infty} E_m^{[\nu,r]}x^k = 0$ for all $k \in \mathbf{N}_0$ and the theorem follows at once. ♠

4 Cauchy principal value integrals

Although the Euler-Maclaurin sum is mostly associated with the evaluation of ordinary integrals, nevertheless it can be used very effectively for the approximate evaluation of Cauchy principal value and certain Hadamard finite-part integrals. In this section we shall consider Cauchy principal value integrals and we shall first introduce the analysis due to Lyness [8]. Let

$$(I_1 f)(x) := \int_0^1 \frac{f(y)}{y-x}dy\,,\quad \text{where } 0 < x < 1\,. \tag{49}$$

To apply Theorem 2.1 we introduce a *subtraction function* ψ which is defined by

$$\psi\left(y;x\right) := \begin{cases} \pi f\left(x\right)\cot\left(\pi\left(y-x\right)\right) , & y-x\notin \mathbf{Z} , \\ 0 , & y-x\in\mathbf{Z} . \end{cases} \tag{50}$$

We now define

$$\left(S_{1,m}^{[\nu]}f\right)\left(x\right) := \begin{cases} \pi f\left(x\right)\cot\left(\pi\left(t_\nu - mx\right)\right) , & t_\nu - mx\notin\mathbf{Z} , \\ -f'\left(x\right)/m , & t_\nu - mx\in\mathbf{Z} , \end{cases} \tag{51}$$

and

$$\left(Q_{1,m}^{[\nu]}f\right)\left(x\right) := \begin{cases} \dfrac{1}{m}\displaystyle\sum_{j=0}^{m-1}\frac{f\left(\left(j+t_\nu\right)/m\right)}{\left(\left(j+t_\nu\right)/m-x\right)} , & t_\nu - mx\notin\mathbf{Z} , \\ \dfrac{1}{m}\displaystyle{\sum_{j=0}^{m-1}}'\frac{f\left(\left(j+t_\nu\right)/m\right)}{\left(\left(j+t_\nu\right)/m-x\right)} , & t_\nu - mx\in\mathbf{Z} , \end{cases} \tag{52}$$

$\sum'$ denoting a sum where the quotient is replaced by zero in that term for which $\left(j+t_\nu\right)/m - x = 0$. We shall now state and prove what is essentially Lyness' principal result [8] noting that the conditions implied in his Theorem 3.5 are slightly incorrect.

Theorem 4.1 *For some $n\in\mathbf{N}$, suppose that $f\in C^{(n-1)}\left[0,1\right]$ with $f^{(n)}$ and $f^{(n+1)}$ continuous on $(0,1)$ and $f^{(n)}$ integrable on $(0,1)$. Then, for $0<x<1$,*

$$\left(I_1 f\right)\left(x\right) = \left(Q_{1,m}^{[\nu]}f\right)\left(x\right) - \left(S_{1,m}^{[\nu]}f\right)\left(x\right)$$

$$- \sum_{j=1}^{n} \frac{\bar{B}_j(t_\nu)}{j!} \frac{1}{m^j} \left\{ \frac{d^{j-1}}{dy^{j-1}} \left(\frac{f(y)}{y-x} \right) \Bigg|_{y=1} - \frac{d^{j-1}}{dy^{j-1}} \left(\frac{f(y)}{y-x} \right) \Bigg|_{y=0} \right\}$$
$$+ \left(E_{1,m}^{[v]} f \right)(x) \tag{53}$$

where

$$\left(E_{1,m}^{[\nu]} f \right)(x) = \frac{1}{m^n} \int_0^1 \frac{d^n}{dy^n} \left(\frac{f(y)}{y-x} - \psi(y;x) \right) \frac{\bar{B}_n(t_\nu - my)}{n!} dy \,. \tag{54}$$

Proof. This is based on applying Theorem 2.1 to the function h_1 defined on $[0,1]$ by

$$h_1(y;x) = \begin{cases} f(y)/(y-x) - \psi(y;x) \,, & y \neq x \,, \\ f'(x) \,, & y = x \,. \end{cases} \tag{55}$$

We shall show that, after suitably defining $\frac{\partial^j h_1}{\partial y^j}(x;x)$ for $j = 1\,(1)\,n$, the conditions on f imply that, for a fixed $x \in (0,1)$, $h_1 \in C^{(n-1)}[0,1]$ and $h_1^{(n)} \in L_1(0,1)$. With this established we can immediately apply Theorem 2.1 to the function h_1 and this will give (53) and (54).

From Abramowitz and Stegun [1, 4.3.17] for $y \in [0,1] \backslash x$

$$\cot(\pi(y-x)) = \frac{1}{\pi(y-x)} - \frac{1}{\pi}\phi_0(y;x) \tag{56}$$

say, where

$$\phi_0(y;x) = \sum_{j=1}^{\infty} \frac{(2\pi)^{2j} |B_{2j}|}{(2j)!} (y-x)^{2j-1} \quad \text{for } |y-x| < 1 \,. \tag{57}$$

We see that $\phi_0(\cdot\,,x) \in C^\infty[0,1]$. From (55)–(57) we have that

$$h_1(y;x) = \frac{f(y)-f(x)}{y-x} + f(x)\,\phi_0(y;x)\ , \quad y \neq x\,. \tag{58}$$

If we define $h_1(x;x)$ to be $\lim_{y\to x} h_1(y;x)$ then from (58) we have $h_1(x;x) = f'(x)$, which agrees with (55), and the function $h_1(\cdot\,;x)$ will be continuous on $[0,1]$ since both f and $\phi_0(\cdot\,;x)$ are continuous on $[0,1]$. If we let $h_1^{(j)}(\cdot\,;x)$ denote the jth order partial derivative of h_1 with respect to y for a given $x \in (0,1)$, we see from (58) that $h_1^{(j)}(\cdot\,;x)$ exists on $[0,1]\setminus x$ for $j = 0\,(1)\,(n-1)$. If we define $h_1^{(j)}(x;x) = \lim_{y\to x} h_1^{(j)}(y;x)$ then $h_1^{(j)}(\cdot\,;x)$ will be continuous at x. Since we are assuming that $f \in C^{(n-1)}[0,1]$ we shall then have $h_1(\cdot\,;x) \in C^{(n-1)}[0,1]$. What is $h_1^{(j)}(x\,;x)$? A slightly tedious calculation gives

$$h_1^{(j)}(x;x) = \begin{cases} f^{(j+1)}(x)\,/\,(j+1)\ , & j \text{ even,} \\ \left[f^{(j+1)}(x) + (2\pi)^{j+1}\left|B_{j+1}\right| f(x)\right]/\,(j+1)\ , & j \text{ odd.} \end{cases} \tag{59}$$

This will be valid for $j = 0\,(1)\,n$ since, when $j = n$, we are assuming that $f^{(n+1)} \in C(0,1)$ and, of course, $x \in (0,1)$.

It remains to show that $h_1^{(n)}(\cdot\,;x)$ is integrable over $(0,1)$. From the preceding paragraph we have that $h_1^{(n)}(\cdot\,;x) \in C(0,1)$ so that integrability over $(0,1)$ will follow if $h_1^{(n)}$ is 'well behaved' at the end points 0 and 1. By Leibnitz' theorem, from (58) we have

$$h_1^{(n)}(y;x) \;=\; (-1)^n\,n!\,(f(y)-f(x))\,/\,(y-x)^{n+1}$$

$$+\sum_{s=1}^{n-1}\binom{n}{s}(-1)^{n-s}(n-s)!f^{(s)}(y)/(y-x)^{n-s+1}$$
$$+f^{(n)}(y)/(y-x)\ . \quad (60)$$

Since $f\in C^{(n-1)}[0,1]$, $x\in(0,1)$ and $f^{(n)}$ is integrable on $(0,1)$ we see that $h_1^{(n)}(y;x)$ must be integrable in the neighbourhoods of the end-points 0 and 1. Consequently $h_1(\cdot\,;x)$ satisfies the conditions of Theorem 2.1 and (53) and (54) follow from (52) on using the properties of the subtraction function $\psi(\cdot\,;x)$ as given in Lyness [8, §2]. ♠

Let us now consider this result given that f is in the space K_α^N; cf. Theorem 3.3.

Theorem 4.2 *Suppose $f\in K_\alpha^N$ where $n<\alpha<n+1$ for some $n\in\mathbf{N}$. Then, for a given $x\in(0,1)$,*

$$(I_1 f)(x)=\left(Q_{1,m}^{[\nu]}f\right)(x)-\left(S_{1,m}^{[\nu]}f\right)(x)+\left(E_{1,m}^{[\nu]}f\right)(x)\ , \quad (61)$$

and there exists a positive constant c, independent of m and x, such that

$$\left|\left(E_{1,m}^{[\nu]}f\right)(x)\right|\le\frac{c\,\|f\|_{\alpha,N}}{(x(1-x))^{n+1-\alpha}\,m^n}\ . \quad (62)$$

Proof. Since $f\in K_\alpha^N$ where $n<\alpha<n+1$, it follows from Definition 3.1(ii) that

$$f^{(j)}(0)=f^{(j)}(1)=0\,,\quad \text{for } j=0\,(1)\,(n-1)\ , \quad (63)$$

so that $f \in C^{(n-1)}[0,1]$. From Definition 3.1(i) we have that both $f^{(n)}$ and $f^{(n+1)}$ are in $C(0,1)$. Finally, from Theorem 3.2(i) we have that $f^{(n)} \in L_1(0,1)$ so that all the conditions on f in Theorem 4.1 are satisfied. From (63) we see that the summation on the right hand side of (53) is zero so that $(I_1 f)(x)$ is of the form given by equation (61). It remains to put a bound on $\left|\left(E_{1,m}^{[\nu]} f\right)(x)\right|$. With $h_1(\cdot\,;x)$ as defined by (55) and recalling that $\left|\bar{B}_n(t_\nu - my)/n!\right|$ is bounded above by $2\zeta(n)/(2\pi)^n$ (see proof of Theorem 3.3) we have from (54) that

$$\left|\left(E_{1,m}^{[\nu]} f\right)(x)\right| \leq \frac{c}{m^n} \int_0^1 \left|h_1^{(n)}(y;x)\right| dy\,. \tag{64}$$

From the form of $h_1(\cdot\,;x)$ as given in (58), since we can write

$$f(x) = f(y) - (y-x)\int_0^1 f'(x+(y-x)t)\,dt\,,$$

we may rewrite (58) as

$$h_1(y;x) = \phi_1(y;x)\int_0^1 f'(x+(y-x)t)\,dt + \phi_0(y;x)\,f(y)\,, \tag{65}$$

where

$$\phi_1(y;x) = 1-(y-x)\,\phi_0(y;x) = \pi(y-x)\cot(\pi(y-x))\,. \tag{66}$$

We observe that both $\phi_0(\cdot\,;x)$ and $\phi_1(\cdot\,;x)$ are in $C^{(\infty)}[0,1]$. By Leibnitz' theorem

$$h_1^{(n)}(y;x) \;=\; \sum_{s=0}^{n} \binom{n}{s} \left\{ \phi_1^{(n-s)}(y;x)\int_0^1 t^s f^{(s+1)}(x+(y-x)t)\,dt \right.$$

$$+\phi_0^{(n-s)}(y;x)\, f^{(s)}(y) \quad \Big\} \,. \tag{67}$$

Now we can find a positive constant c such that

$$\max_{0\le y\le 1,\ s=0(1)n} \left|\phi_i^{(n-s)}(y;x)\right| \le c\,, \quad \text{for } i=0 \text{ and } 1\,. \tag{68}$$

Consequently from (67) and (68) we find

$$\begin{aligned}\int_0^1 \left|h_1^{(n)}(y;x)\right| dy \;\; &\le \;\; c\sum_{s=0}^{n} \binom{n}{s} \left\{ \int_0^1 \left|f^{(s)}(y)\right| dy \right.\\ &\left. + \int_0^1\int_0^1 t^s \left|f^{(s+1)}(x+(y-x)\,t)\right| dt\, dy \right\}. \end{aligned}\tag{69}$$

By Theorem 3.2(i) we have

$$\int_0^1 \left|f^{(s)}(y)\right| dy \le c\,\|f\|_{\alpha,N} \tag{70}$$

for some positive constant c. For $s = 0\,(1)\,n$, let

$$\begin{aligned} I_s(x) \;\; &:= \;\; \int_0^1\int_0^1 t^s \left|f^{(s+1)}(x+(y-x)\,t)\right| dt\, dy \\ &= \;\; \int_0^1 t^s \int_0^1 \left|f^{(s+1)}(x+(y-x)\,t)\right| dy\, dt\,. \end{aligned}\tag{71}$$

Since $0 < x < 1$ and $0 \le t, y \le 1$ we have $0 \le x + (y-x)\,t \le 1$ so that we shall have

$$\int_0^1 \left|f^{(s+1)}(x+(y-x)\,t)\right| dy \le \int_0^1 \left|f^{(s+1)}(u)\right| du \le c\,\|f\|_{\alpha,N} \tag{72}$$

for $s = 0\,(1)\,(n-1)$ by Theorem 3.2(i). Thus for $s = 0\,(1)\,(n-1)$ we have

$$\int_0^1 \int_0^1 t^s \left| f^{(s+1)}\left(x + (y-x)\,t\right) \right| dt\, dy \le c\, \|f\|_{\alpha,N} \tag{73}$$

for some positive constant c. It remains to put an upper bound on $I_n(x)$. From (71), on putting $u = x + (y-x)\,t$ we have

$$I_n(x) = \int_0^1 t^{n-1} J_n(t;x)\, dt \tag{74}$$

say, where

$$J_n(t;x) := \int_{x(1-t)}^{t+x(1-t)} \left| f^{(n+1)}(u) \right| du\,. \tag{75}$$

Now we can write

$$\begin{aligned} J_n(t;x) &= \int_{x(1-t)}^{t+x(1-t)} \left(u\,(1-u)\right)^{n+1-\alpha} \left| f^{(n+1)}(u) \right| \cdot \left(u\,(1-u)\right)^{\alpha-(n+1)} du \\ &\le H_{\alpha,n+1}(x;t)\, \|f\|_{\alpha,N} \end{aligned} \tag{76}$$

say, where

$$\begin{aligned} H_{\alpha,n+1}(x;t) &:= \max_{x(1-t)\le u\le t+x(1-t)} \left(u\,(1-u)\right)^{\alpha-(n+1)} \\ &= \frac{1}{\min\limits_{0\le v\le 1} \left(Q\,(v;x,t)\right)^{n+1-\alpha}} \end{aligned} \tag{77}$$

since $\alpha < n+1$, where we define

$$Q\,(v;x,t) := \left(x\,(1-t) + tv\right)\left(1 - x\,(1-t) - tv\right)\,. \tag{78}$$

To determine the minimum value of Q for $v \in [0,1]$ we shall distinguish two cases. Suppose first that $0 < x \leq 1/2$. Since $0 \leq t, v \leq 1$ we have

$$t\,(v - 2x) \leq v - 2x \leq 1 - 2x$$

so that

$$0 \leq 1 - 2x\,(1-t) - vt$$

or, on multiplying by vt, which is non-negative,

$$0 \leq vt\,(1 - 2x\,(1-t)) - v^2 t^2\,. \tag{79}$$

Combining (78) and (79) we have

$$Q\,(0; x, t) \leq Q\,(0; x, t) + vt\,(1 - 2x\,(1-t)) - v^2 t^2 = Q\,(v; x, t)\,. \tag{80}$$

Hence, for $0 < x \leq 1/2$

$$\begin{aligned} H_{\alpha,n+1}\,(x; t) &= 1/\,(Q\,(0; x, t))^{n+1-\alpha} \\ &= 1/\left[(x\,(1-t))^{n+1-\alpha}\,(1 - x\,(1-t))^{n+1-\alpha}\right] \\ &\leq 2^{n+1-\alpha}/\,(x\,(1-t))^{n+1-\alpha}\,, \end{aligned} \tag{81}$$

since $1 - x\,(1-t) \geq 1/2$ for $t \in [0,1]$. From (74), (76) and (81) we find for $x \in (0, 1/2]$,

$$I_n\,(x) \leq 2^{n+1-\alpha}\,\|f\|_{\alpha,N}\,(\Gamma\,(n)\,\Gamma\,(\alpha - n)\,/\Gamma\,(\alpha))\,/x^{n+1-\alpha}\,. \tag{82}$$

We may argue similarly for $x \in [1/2, 1)$. We shall now have $Q(v; x, t) \geq Q(1; x, t) \geq (1-x)(1-t)/2$ for all $t \in [0,1]$. Consequently for $1/2 \leq x < 1$,

$$I_n(x) \leq 2^{n+1-\alpha} \|f\|_{\alpha,N} \left(\Gamma(n)\Gamma(\alpha - n)/\Gamma(\alpha)\right) / (1-x)^{n+1-\alpha} . \tag{83}$$

Combining (82) and (83) we conclude that for any given $x \in (0,1)$ there exists a positive constant c such that

$$I_n(x) \leq c \|f\|_{\alpha,N} / (x(1-x))^{n+1-\alpha} . \tag{84}$$

Returning to (69), recalling (70) and (73), together with (84) we conclude that there exists a positive constant c independent of n and x such that

$$\int_0^1 \left| h_1^{(n)}(y; x) \right| dy \leq c \|f\|_{\alpha,N} / (x(1-x))^{n+1-\alpha} \tag{85}$$

for a given $x \in (0,1)$. Combining this with (64) establishes (62) and the theorem is proved. ♠

We must now consider the effect of a sigmoidal transformation of order r applied to the integral. On writing $y = \gamma_r(t)$ then, as with the change of variable for ordinary integrals, we have

$$(I_1 f)(x) = \int_0^1 \frac{\gamma_r'(t) f(\gamma_r(t))}{\gamma_r(t) - x} dt . \tag{86}$$

If we write

$$x = \gamma_r(s) \tag{87}$$

then, since $x \in (0,1)$, we shall have $s \in (0,1)$. Consequently we shall rewrite $(I_1 f)(x)$ as

$$(I_1 f)(x) = \int_0^1 \frac{\Phi_r(t;s)}{t-s} dt \tag{88}$$

say, where

$$\Phi_r(t;s) := \begin{cases} \frac{\gamma_r'(t)(t-s)f(\gamma_r(t))}{\gamma_r(t)-\gamma_r(s)}, & t \neq s, \\ f(x), & t = s. \end{cases} \tag{89}$$

The question now arises as to which space $\Phi_r(\cdot\,;s)$ is in given that $f \in K_\alpha^N$ for some $\alpha > 0$, and we shall address this in the next theorem.

Theorem 4.3 *Suppose $f \in K_\alpha^N$ for some non-integer $\alpha > 0$ and let γ_r be a sigmoidal transformation of order $r \geq 1$. Suppose $\beta = \alpha r$ with $\beta \notin$* **N**. *Assuming that $\beta < N$ then*

(i) $\Phi_r(\cdot\,;s) \in K_\beta^N$,

(ii) there exists a positive constant c such that

$$\|\Phi_r(\cdot\,;s)\|_{\beta,N} \leq c\,\|f\|_{\alpha,N}\,.$$

Proof. Let us write

$$\Phi_r(t;s) = \rho_r(t;s)\,g_r(t) \tag{90}$$

say, where g_r is defined in (33) and ρ_r is defined by

$$\rho_r(t;s) := \begin{cases} (t-s)/(\gamma_r(t)-\gamma_r(s)), & t \neq s, \\ 1/\gamma_r'(s), & t = s. \end{cases} \tag{91}$$

Since s is fixed in $(0,1)$ and $\rho_r(s;s) = \lim_{t\to s} \rho_r(t;s)$ we see that $\rho_r(\cdot\,;s)$ is continuous on $[0,1]$. However, we can say more. If, for all $j \in \mathbf{N}$, we define $\rho_r^{(j)}(s;s) := \lim_{t\to s} \rho_r^{(j)}(t;s)$ then we shall have $\rho_r(\cdot\,;s) \in C^{(\infty)}[0,1]$. It should be noted that we can define $\rho_r^{(j)}(\cdot\,;s)$ recursively. From (91) we can write

$$\rho_r(t;s) \int_0^1 \gamma_r'(s+(t-s)\eta)\, d\eta = 1 \tag{92}$$

for $0 \le s, t \le 1$. With fixed $s \in (0,1)$, on differentiating j times with respect to t and putting $t = s$ we obtain

$$\rho_r^{(j)}(s;s) = -\frac{1}{(j+1)\gamma_r'(s)} \sum_{i=0}^{j-1} \binom{j+1}{i} \rho_r^{(i)}(s;s)\, \gamma_r^{(j+1-i)}(s) \tag{93}$$

for $j \in \mathbf{N}$ with $\rho_r(s;s) = 1/\gamma_r'(s)$. Since $\gamma_r'(s) \neq 0$ we see that $\rho_r^{(j)}(s;s)$ is defined for all $j \in \mathbf{N}$. From Theorem 3.5(i), $g_r \in K_\beta^N$ so that since $g_r^{(j)} \in C(0,1)$ for $j = 0\,(1)\,N$, it follows that for a given $s \in (0,1)$, $\Phi_r^{(j)}(\cdot\,;s) \in C(0,1)$ for $j = 0\,(1)\,N$. This is condition (i) of Definition 3.1.

Near $t = 0$, see that $\rho_r(t;s) = (s/\gamma_r(s))(1+O(t))$, the constant $(s/\gamma_r(s))$ being defined and non-zero since $s \in (0,1)$. Thus, near $t = 0$, $\Phi_r(\cdot\,;s)$ behaves like g_r and we know from (41) that $g_r^{(j)}(t) = O\left(t^{\alpha r - j - 1}\right)$, for $j = 0\,(1)\,(N-1)$.

With a similar result near $t = 1$ we find that if $n_1 < \beta < n_1 + 1$, for some $n_1 \in \mathbf{N}$ (recall that $\beta = \alpha r$ is not an integer) then we shall have

$$\Phi_r^{(j)}(0; s) = \Phi_r^{(j)}(1; s) = 0 \quad \text{for } j = 0\,(1)\,(n_1 - 1)\ . \tag{94}$$

This satisfies Definition 3.1(ii).

Finally, from (90), for a fixed $s \in (0, 1)$ we have, by Leibnitz's theorem, that for $j = 0\,(1)\,N$

$$\Phi_r^{(j)}(t; s) = \sum_{i=0}^{j} \binom{j}{i} g_r^{(i)}(t)\, \rho_r^{(j-i)}(t; s) \tag{95}$$

so that

$$\begin{aligned}\int_0^1 (t\,(1-t))^{j-\beta} \left|\Phi_r^{(j)}(t; s)\right| dt \;\; &\le \;\; \sum_{i=0}^{j} \binom{j}{i} \int_0^1 (t\,(1-t))^{i-\beta} \left|g_r^{(i)}(t)\right| \times \\ &\qquad (t\,(1-t))^{j-i} \left|\rho_r^{(j-i)}(t; s)\right| dt\,. \end{aligned} \tag{96}$$

Now, since $\rho_r(\cdot\,, s) \in C^\infty[0, 1]$ we can find a positive constant c such that

$$\max_{0 \le t \le 1} \left|\rho_r^{(j-i)}(t; s)\right| \le c \quad \text{for } i = 0\,(1)\,j \text{ and } j = 0\,(1)\,N\,.$$

Again, since $j - i \ge 0$ we can find a positive constant c such that from (96)

$$\int_0^1 (t\,(1-t))^{j-\beta} \left|\Phi_r^{(j)}(t; s)\right| dt \le c\,\|g_r\|_{\beta,N} \sum_{i=0}^{j} \binom{j}{i} < \infty\,. \tag{97}$$

Thus from Definition 3.1(iii) we see that $\Phi_r \in K_\beta^N$ which proves (i). Furthermore, (97) tells us that there exists a positive constant c such that $\|\Phi_r\|_{\beta,N} \le c\,\|g_r\|_{\beta,N}$ which completes the proof of the theorem. ♠

We now come to the approximate evaluation of $(I_1 f)(x)$ from the form given by (86). To this end we now define the following quantities

$$\left(Q_{1,m}^{[\nu,r]} f\right)(x) := \begin{cases} \dfrac{1}{m} \displaystyle\sum_{j=0}^{m-1} \dfrac{\gamma_r'\left((j+t_\nu)/m\right) f\left(\gamma_r((j+t_\nu)/m)\right)}{\left(\gamma_r\left((j+t_\nu)/m\right) - \gamma_r(s)\right)}\,, & ms - t_\nu \notin \mathbf{Z}\,, \\ \dfrac{1}{m} \displaystyle{\sum_{j=0}^{m-1}}' \dfrac{\gamma_r'\left((j+t_\nu)/m\right) f\left(\gamma_r((j+t_\nu)/m)\right)}{\left(\gamma_r\left((j+t_\nu)/m\right) - \gamma_r(s)\right)}\,, & ms - t_\nu \in \mathbf{Z}\,, \end{cases} \tag{98}$$

with $\sum'$ denoting, as before, a sum where the term in which the denominator is zero is replaced by zero. From (89) we observe that

$$\left(Q_{1,m}^{[\nu,r]} f\right)(x) = \left(Q_{1,m}^{[\nu]} \Phi_r\right)(s)\ . \tag{99}$$

In addition, we define

$$\left(S_{1,m}^{[\nu,r]} f\right)(x) := \begin{cases} \pi f(x) \cot\left(\pi\left(t_\nu - ms\right)\right)\,, & ms - t_\nu \notin \mathbf{Z} \\ -\frac{\gamma_r''(s) f(x)}{2m\gamma_r'(s)} + \gamma_r'(s)\, f'(x)\,, & ms - t_\nu \in \mathbf{Z}\,. \end{cases} \tag{100}$$

Again, we see that

$$\left(S_{1,m}^{[\nu,r]} f\right)(x) = \left(S_{1,m}^{[\nu]} \Phi_r\right)(s)\ . \tag{101}$$

We now come to the principal result of this section.

Theorem 4.4 *Suppose* $f \in K_\alpha^N$ *for some non-integer* $\alpha > 0$. *Let* γ_r *be a sigmoidal transformation of order* $r \geq 1$ *such that* $n_1 < \alpha r < n_1 + 1$ *for some* $n_1 \in \mathbf{N}$. *Then, for any* $m \in \mathbf{N}$ *and* $0 < x < 1$,

$$\left(I_1 f\right)(x) = \left(Q_{1,m}^{[\nu,r]} f\right)(x) - \left(S_{1,m}^{[\nu,r]} f\right)(x) + \left(E_{1,m}^{[\nu,r]} f\right)(x) \tag{102}$$

say, where

$$\left|\left(E_{1,m}^{[\nu,r]} f\right)(x)\right| \leq \frac{c\left\|f\right\|_{\alpha,N}}{m^{n_1}\left(x\left(1-x\right)\right)^{(n_1+1-\alpha r)/r}}\,, \tag{103}$$

for some positive constant c *independent of* m *and* x.

Proof. From (61), (99) and (101) we have

$$\left|\left(E_{1,m}^{[\nu,r]} f\right)(x)\right| = \left|\left(E_{1,m}^{[\nu]} \Phi_r\right)(s)\right|\,. \tag{104}$$

Since by Theorem 4.3, $\Phi_r \in K_\beta^N$ where $\beta = \alpha r$, then on using (62) we have

$$\begin{aligned} \left|\left(E_{1,m}^{[\nu]} \Phi_r\right)(s)\right| &\leq \frac{c\left\|\Phi_r\right\|_{\beta,N}}{\left(s\left(1-s\right)\right)^{n_1+1-\beta} m^{n_1}} \\ &\leq \frac{c\left\|f\right\|_{\alpha,N}}{\left(\gamma_r^{-1}(x)\left(1-\gamma_r^{-1}(x)\right)\right)^{n_1+1-\beta} m^{n_1}}\,, \end{aligned} \tag{105}$$

by Theorem 4.2 for some positive constant c. Now near $x = 0$, $\gamma_r^{-1}(x) = O\left(x^{1/r}\right)$ and near $x = 1$ we have $1 - \gamma_r^{-1}(x) = O\left((1-x)^{1/r}\right)$. Consequently we may write

$$\gamma_r^{-1}(x)\left(1-\gamma_r^{-1}(x)\right) = \left(x\left(1-x\right)\right)^{1/r} X_r(x) \tag{106}$$

say, where the function X_r is continuous and strictly positive on $[0,1]$. Combining (105) with (106) gives (103) and the theorem is proved. ♠

We see that as in the case of the ordinary integral (see Theorem 3.6) the introduction of a sigmoidal transformation or order r increases the rate of convergence from $O\left(1/m^n\right)$ to $O\left(1/m^{n_1}\right)$ where n_1 is roughly r times n. This can be very satisfactory (see Elliott & Venturino [5]). Furthermore it is of interest to know from the above analysis that the rates of convergence are the same for both ordinary and Cauchy principal value integral. However, in the latter case the presence of the factor $1/\left(x\left(1-x\right)\right)^{(n_1+1-\alpha r)/r}$ indicates that the actual values of the errors will be larger near the end-points. This is well borne out in practice.

In Elliott & Venturino [5], we considered the use of the Euler-Maclaurin summation together with sigmoidal transformations to find approximate values of certain Hadamard finite-part integrals. We shall discuss such integrals in the next section.

5 Hadamard finite-part integrals

We now turn our attention to the particular Hadamard finite-part integral $\left(I_2 f\right)(x)$ which is given by

$$\left(I_2 f\right)(x) := \int_0^1 \frac{f\left(y\right)}{\left(y-x\right)^2}\,dy := \frac{d}{dx}\left\{\int_0^1 \frac{f\left(y\right)}{\left(y-x\right)}\,dy\right\}, \qquad (107)$$

for some $x \in (0,1)$. Since it is essentially the derivative of the Cauchy principal value integral $I_1 f$, all our results will follow on differentiation with respect to x. From (98) and (100) we shall define, for $ms - t_\nu \notin \mathbf{Z}$,

$$\left(Q_{2,m}^{[\nu,r]} f\right)(x) = \frac{1}{m} \sum_{j=0}^{m-1} \frac{\gamma_r'\left(\left(j+t_\nu\right)/m\right) f\left(\gamma_r\left(\left(j+t_\nu\right)/m\right)\right)}{\left(\gamma_r\left(\left(j+t_\nu\right)/m\right) - \gamma_r(s)\right)^2} \tag{108}$$

and

$$\left(S_{2,m}^{[\nu,r]} f\right) = \pi f'\left(\gamma_r(s)\right) \cot\left(\pi\left(t_\nu - ms\right)\right) + \pi^2 m \frac{f\left(\gamma_r(s)\right)}{\gamma_r'(s)} \operatorname{cosec}^2\left(\pi\left(t_\nu - ms\right)\right), \tag{109}$$

where $x = \gamma_r(s)$. Thus after sigmoidal transformation we shall write

$$\left(I_2 f\right)(x) = \left(Q_{2,m}^{[\nu,r]} f\right)(x) - \left(S_{2,m}^{[\nu,r]} f\right)(x) + \left(E_{2,m}^{[\nu,r]} f\right)(x) \tag{110}$$

say, where we need to put a bound on $\left|\left(E_{2,m}^{[\nu,r]} f\right)(x)\right|$. The first two terms on the right-hand side of (110) give the approximation to $\left(I_2 f\right)(x)$; we want to bound the "error term". We might write the corresponding forms for $\left(Q_{2,m}^{[\nu,r]} f\right)(x)$ and $\left(S_{2,m}^{[\nu,r]} f\right)(x)$ when $ms - t_\nu \in \mathbf{Z}$, but the details will be omitted here.

We shall now develop the analysis for $\left(I_2 f\right)(x)$ in a manner similar to that for $\left(I_1 f\right)(x)$. In (108) and (109) we shall adopt the convention that when $r = 1$ then $\gamma_1(x) = x$ so that we shall write $\left(Q_{2,m}^{[\nu,1]} f\right)(x)$ as $\left(Q_{2,m}^{[\nu]} f\right)(x)$ which is equal to $\frac{d}{dx}\left(Q_{1,m}^{[\nu,1]} f\right)(x)$, see (52). A similar remark applies to $\left(S_{2,m}^{[\nu,1]} f\right)(x)$.

Theorem 5.1 *For some* $n \in \mathbf{N}$, *suppose that* $f \in C^{(n-1)}[0,1]$, *with* $f^{(n)}$, $f^{(n+1)}$ *and* $f^{(n+2)}$ *continuous on* $(0,1)$. *Suppose in addition that* $f^{(n)}$ *is integrable on* $(0,1)$. *Then for some* $x \in (0,1)$

$$\begin{aligned} & (I_2 f)(x) \\ = \; & \left(Q_{2,m}^{[\nu]} f\right)(x) - \left(S_{2,m}^{[\nu]} f\right)(x) \\ & - \sum_{j=1}^{n} \frac{\bar{B}_j(t_\nu)}{j!} \frac{1}{m^j} \left\{ \frac{d^{j-1}}{dy^{j-1}} \left(\frac{f(y)}{(y-x)^2} \right) \Bigg|_{y=1} - \frac{d^{j-1}}{dy^{j-1}} \left(\frac{f(y)}{(y-x)^2} \right) \Bigg|_{y=0} \right\} \\ & + \left(E_{2,m}^{[\nu]} f\right)(x) \end{aligned} \qquad (111)$$

where

$$\left(E_{2,m}^{[\nu]} f\right)(x) = \frac{1}{m^n} \int_0^1 \frac{d^n}{dy^n} \left(\frac{f(y)}{(y-x)^2} - \frac{\partial \psi(y;x)}{\partial x} \right) \frac{\bar{B}_n(t_\nu - my)}{n!} dy \,. \qquad (112)$$

Proof. This parallels that of Theorem 4.1. For $y \neq x$ we define $h_2(y;x)$ to be $\frac{\partial}{\partial x} h_1(y;x)$ (see equation (55)). The value of $h_2(x;x)$ is defined to be $\lim_{y \to x} h_2(y;x)$. Consequently we define

$$h_2(y;x) := \begin{cases} f(y)/(y-x)^2 - \frac{\partial \psi}{\partial x}(y;x) \,, & y \neq x \,, \\ \frac{1}{2} f''(x) - \frac{\pi^2}{3} f(x) \,, & y = x \,. \end{cases} \qquad (113)$$

It is to the function h_2 that we shall apply Theorem 2.1. We need to show that after suitably defining $\frac{\partial^j h_2}{\partial y^j}(x;x)$ for $j = 1\,(1)\,n$, the conditions on f imply that

$h_2 \in C^{(n-1)}[0,1]$ and $h_2^{(n)} \in L_1(0,1)$. If we take the expression for h_1 as given by (65) then we have

$$\begin{aligned} h_2(y;x) &= \frac{\partial}{\partial x} h_1(y;x) \\ &= \frac{\partial \phi_1}{\partial x}(y;x) \int_0^1 f'(x+(y-x)t)\,dt \\ &\quad + \phi_1(y;x) \int_0^1 (1-t) f''(x+(y-x)t)\,dt \\ &\quad + \frac{\partial \phi_0}{\partial x}(y;x) f(y) , \end{aligned} \tag{114}$$

where ϕ_0 and ϕ_1 are defined by (57) and (66) respectively. A tedious calculation shows that

$$h_2^{(j)}(x;x) = \begin{cases} \frac{f^{(j+2)}(x)}{(j+1)(j+2)} - (2\pi)^{j+2} |B_{j+2}| \frac{f(x)}{(j+2)}, & j \text{ even,} \\ \frac{f^{(j+2)}(x)}{(j+1)(j+2)} + (2\pi)^{j+1} |B_{j+1}| \frac{f'(x)}{(j+1)}, & j \text{ odd,} \end{cases} \tag{115}$$

where we define $h_2^{(j)}(x;x)$ to be $\lim_{y \to x} h_2^{(j)}(y;x)$. This will be valid for $j = 0\,(1)\,n$ since when $j = n$ we are assuming that both $f^{(n+1)}$ and $f^{(n+2)}$ are continuous at $x \in (0,1)$.

For the integrability of $h_2^{(n)}(\cdot\,;x)$ over $(0,1)$ we argue as for the integrability of $h_1^{(n)}(\cdot\,;x)$ in the proof of Theorem 4.1. We have $h_2^{(n)}(\cdot\,;x) \in C(0,1)$ so that we need only consider h_2 at the end points 0 and 1. As with h_1 in Theorem 4.1, we may consider $h_2^{(n)}(y;x)$ by applying Leibnitz's theorem to each term in (114). The behaviour of f and its derivatives at the end points 0 and 1 show that $h_2^{(n)}(\cdot\,;x)$ is integrable on $(0,1)$. The details are left to the reader. ♠

As in §4, let us now assume that $f \in K_\alpha^N$ for some non-integer $\alpha > 0$.

Theorem 5.2 *Suppose $f \in K_\alpha^N$ where $n < \alpha < n+1$ for some $n \in \mathbf{N}$. Then, for a given $x \in (0,1)$*

$$(I_2 f)(x) = \left(Q_{2,m}^{[\nu]} f\right)(x) - \left(S_{2,m}^{[\nu]} f\right)(x) + \left(E_{2,m}^{[\nu]} f\right)(x)\,, \tag{116}$$

and there exists a positive constant c, independent of m and x, such that

$$\left|\left(E_{2,m}^{[\nu]} f\right)(x)\right| \le \frac{c\,\|f\|_{\alpha,N}}{\left(x\,(1-x)\right)^{n+2-\alpha}\, m^n}\,. \tag{117}$$

Proof. That the form of $(I_2 f)(x)$ is as given in (116) follows by the same arguments as given in the proof of Theorem 4.2. From (112) we have that

$$\left|\left(E_{2,m}^{[\nu]} f\right)(x)\right| < \frac{c}{m^n} \int_0^1 \left|h_2^{(n)}(y;x)\right| dy\,.$$

From the form of h_2 as given by (114) we find on recalling (68)

$$\int_0^1 \left|h_2^{(n)}(y;x)\right| dx \;\le\; c\sum_{s=0}^{n} \binom{n}{s} \left\{ \int_0^1 \int_0^1 t^s \left| f^{(s+1)}(x+(y-x)\,t)\right| dt\, dy \right.$$

$$+ \int_0^1 \int_0^1 t^s (1-t) \left| f^{(s+2)} (x + (y-x) t) \right| dt\, dy$$
$$\left. + \int_0^1 \left| f^{(s)} (y) \right| dy \right\} . \tag{118}$$

Now (70) gives a bound on the last term on the right hand side of (118). The first term on the right hand side of (118) we have previously denoted by $I_s(x)$ (see (71)) and is bounded above by $c \|f\|_{\alpha,N}$ for $s = 0\,(1)\,(n-1)$ (see (72)). When $s = n$, a bound for $I_n(x)$ is given by (84). Finally let us consider

$$\begin{aligned} K_s(x) &:= \int_0^1 \int_0^1 t^s (1-t) \left| f^{(s+2)} (x + (y-x) t) \right| dt\, dy \\ &= \int_0^1 t^s (1-t) \int_0^1 \left| f^{(s+2)} (x + (y-x) t) \right| dy\, dt , \end{aligned} \tag{119}$$

for $s = 0\,(1)\,n$. For $s = 0\,(1)\,(n-2)$ we have simply (cf. (72))

$$K_s(x) \leq \left(\int_0^1 t^s (1-t)\, dt \right) \left(\int_0^1 \left| f^{(s+2)} (u) \right| du \right) \leq c \|f\|_{\alpha,N} . \tag{120}$$

Now (cf. (74) and (75)) we have

$$K_{n-1}(x) = \int_0^1 t^{n-1} (1-t)\, J_n(t;x)\, dt . \tag{121}$$

For $0 < x \leq 1/2$ we find from (76)–(81) that

$$\begin{aligned} K_{n-1}(x) &\leq \frac{2^{n+1-\alpha}}{x^{n+1-\alpha}} \|f\|_{\alpha,N} \int_0^1 t^{n-1} (1-t)^{\alpha-n}\, dt \\ &\leq c \|f\|_{\alpha,N} / x^{n+1-\alpha} , \end{aligned} \tag{122}$$

for some positive constant c. Arguing similarly for $1/2 \le x < 1$ we conclude that

$$K_{n-1}(x) \le \frac{c\,\|f\|_{\alpha,N}}{(x(1-x))^{n+1-\alpha}}\,, \tag{123}$$

for $0 < x < 1$. Finally, it remains to consider $K_n(x)$. By arguing as we have done for $K_{n-1}(x)$ we find

$$K_n(x) \le \frac{c\,\|f\|_{\alpha,N}}{(x(1-x))^{n+2-\alpha}}\,. \tag{124}$$

Putting these results together we establish (117) and the theorem is proved. ♠

We must now consider the effect of a sigmoidal transformation on the Hadamard finite-part integral. On writing $y = \gamma_r(t)$, $x = \gamma_r(s)$ we find from (107) that we may write

$$(I_2 f)(x) = \int_0^1 \frac{\Psi_r(t;s)}{(t-s)^2}\,dt \tag{125}$$

say, where the function Ψ_r is defined by

$$\Psi_r(t;s) = \begin{cases} f(\gamma_r(t))\,\gamma_r'(t)\,((t-s)/(\gamma_r(t)-\gamma_r(s)))^2\,, & t \ne s\,, \\ f(\gamma_r(s))\,/\gamma_r'(s)\,, & t = s\,. \end{cases} \tag{126}$$

Theorem 5.3 *Suppose $f \in K_N^\alpha$ for some non-integer $\alpha > 0$ and let γ_r be a sigmoidal transformation of order $r \ge 1$. Suppose $\beta = \alpha r$ with $\beta \notin \mathbf{N}$. Assuming $\beta < N$ then* *(i)* $\Psi_r(\cdot\,;s) \in K_\beta^N$,

(ii) there exists a positive constant c *such that*

$$\|\Psi_r(\cdot\,;s)\|_{\beta,N} \le c\,\|f\|_{\alpha,N}\ .$$

Proof. Recalling (33), (90) and (91) we see that we can write

$$\Psi_r(t;s) = \rho_r^2(t;s)\,g_r(t) = \rho_r(t;s)\,\Phi_r(t;s)\ . \tag{127}$$

In the proof of Theorem 4.3 we showed that $\rho_r(\cdot\,;s) \in C^{(\infty)}[0,1]$. Just as in the proof of that theorem, $g_r \in K_\beta^N$ implied $\Phi_r(\cdot\,;s) \in K_\beta^N$ so we shall have $\Psi_r(\cdot\,;s) \in K_\beta^N$ and furthermore $\|\Psi_r(\cdot\,;s)\|_{\beta,N} \le c\,\|f\|_{\alpha,N}$ for some positive constant c. The details will be omitted. ♠

We now come to our principal result for these particular Hadamard finite-part integrals.

Theorem 5.4 *Suppose* $f \in K_\alpha^N$ *for some non-integer* $\alpha > 0$. *Let* γ_r *be a sigmoidal transformation of order* $r \ge 1$ *such that* $n_1 < \alpha r < n_1 + 1$ *for some* $n_1 \in \mathbf{N}$. *Then for any* $m \in \mathbf{N}$ *and* $0 < x < 1$,

$$(I_2 f)(x) = \left(Q_{2,m}^{[\nu,r]} f\right)(x) - \left(S_{2,m}^{[\nu,r]} f\right)(x) + \left(E_{2,m}^{[\nu,r]} f\right)(x) \tag{128}$$

where

$$\left|\left(E_{2,m}^{[\nu,r]} f\right)(x)\right| \le \frac{c\,\|f\|_{\alpha,N}}{m^{n_1}\,(x\,(1-x))^{(n_1+2-\alpha r)/r}}\,, \tag{129}$$

for some positive constant c *independent of* m *and* x.

Proof. As in the proof of Theorem 4.4 we have

$$\begin{aligned} \left|\left(E_{2,m}^{[\nu,r]} f\right)(x)\right| &= \left|\left(E_{2,m}^{[\nu]} \Psi_r\right)(s)\right| \\ &\leq \frac{c\left\|\Psi_r\right\|_{\beta,N}}{\left(s\left(1-s\right)\right)^{n_1+2-\beta} \cdot m^{n_1}}, \quad \text{by Theorem 5.2}, \\ &\leq \frac{c\left\|f\right\|_{\alpha,N}}{\left(\gamma_r^{-1}\left(x\right)\left(1-\gamma_r^{-1}\left(x\right)\right)\right)^{n_1+2-\beta} m^{n_1}}, \quad \text{by Theorem 5.3}, \end{aligned} \tag{130}$$

from which (129) follows, arguing as in the proof of Theorem 4.4.

6 Conclusion

We have considered the application of the Euler-Maclaurin summation formula together with the use of sigmoidal transformations for ordinary, Cauchy principal value and certain Hadamard finite-part integrals over the finite interval $(0,1)$. We have defined a suitable space of functions in which to consider the analysis and, perhaps surprisingly, one finds that the rate of convergence of the error to zero is the same in each case. The analysis we have given here looks as if it should provide a basis for using this particular set of quadrature rules in the approximate solution of both Fredholm and singular integral equations taken over the finite interval $(0,1)$. That investigation however will have to wait for another paper; this one is long enough already.

A Proof of equation (115)

We observe

$$\frac{\partial\phi_0}{\partial x}+\frac{\partial\phi_0}{\partial y}=0 \tag{131}$$

and

$$\frac{\partial\phi_1}{\partial x}+\frac{\partial\phi_1}{\partial y}=0\,. \tag{132}$$

Now $h_2 := \frac{\partial h_1}{\partial x}$ and

$$\begin{aligned}\frac{\partial h_1}{\partial x} &= \frac{\partial\phi_1}{\partial x}\int_0^1 f'\left(x+\left(y-x\right)t\right)dt \\ &\quad + \phi_1\int_0^1 \left(1-t\right)f''\left(x+\left(y-x\right)t\right)dt+\frac{\partial\phi_0}{\partial x}f\left(y\right)\ .\end{aligned}$$

Again,

$$\begin{aligned}\frac{\partial h_1}{\partial y} &= \frac{\partial\phi_1}{\partial y}\int_0^1 f'\left(x+\left(y-x\right)t\right)dt+\phi_1\int_0^1 t\,f''\left(x+\left(y-x\right)t\right)dt \\ &\quad + \frac{\partial\phi_0}{\partial y}f\left(y\right)+\phi_0 f'\left(y\right)\ .\end{aligned}$$

Therefore on using equations (131) and (132) we have

$$\frac{\partial h_1}{\partial x}+\frac{\partial h_1}{\partial y} = \phi_1\int_0^1 f''\left(x+\left(y-x\right)t\right)dt+\phi_0 f'\left(y\right)$$

$$= \left(1-(y-x)\,\phi_0\right)\left(\frac{f'(y)-f'(x)}{y-x}\right)+\phi_0 f'(y)$$
$$= \frac{f'(y)-f'(x)}{y-x}-\pi f'(x)\cot\left(\pi\,(y-x)\right)$$
$$= h_1\left(y;x;f'\right)$$

say if, from (55), we change the notation to $h_1\,(y;x;f)$. Therefore

$$h_2\,(y;x)=\frac{\partial h_1}{\partial x}=-\frac{\partial h_1}{\partial y}+h_1\left(y;x;f'\right)\,.$$

The result for $h_2^{(j)}\,(x;x)$ now follows from (59).

References

[1] Abramowitz, M. and Stegun, I.A.: *Handbook of Mathematical Functions*, Dover, New York, 1965. E30, E31, E52

[2] Davis, P.J.: *Interpolation and Approximation*, Blaisdell, New York and Toronto, 1965. E49

[3] Elliott, D.: Sigmoidal transformations and the trapezoidal rule, *J. Austral. Maths. Soc. Ser. B*, **40**, 1998, pp.E77–E133. [Online] http://jamsb.austms.org.au/V40/E006 [2 Nov 1998]. E30, E43

[4] Elliott, D. and Prössdorf, S.: An algorithm for the approximate solution of integral equations of Mellin type, *Numer. Math.*, **70**, 1995, pp.427–452. E37

[5] Elliott, D. and Venturino, E.: Sigmoidal transformations and the Euler-Maclaurin expansion for evaluating certain Hadamard finite-part integrals, *Numer. Math.*, **77**, 1997, pp.453–465. E30, E65, E65

[6] Gradshteyn, I.S. and Ryzhik, J.M.: *Tables of Integrals, Series and Products*, 5th edition edited by A. Jeffrey, Academic Press, Boston, 1994. E30

[7] Kress, R.: A Nyström method for boundary integral equations in domains with corners, *Numer. Math.* **58**, 1990, pp.145–161. E37

[8] Lyness, J.: The Euler Maclaurin expansion for the Cauchy principal value integral, *Numer. Math.*, **46**, 1985, pp.611–622. E29, E32, E32, E50, E51, E54

[9] Olver, F.W.J.: *Asymptotics and Special Functions*, Academic Press, New York and London, 1974. E30

[10] Steffensen, J.F.: *Interpolation*, Chelsea Publ. Co., New York, 1965. E30, E31