J. Austral. Math. Soc. B 40 (E) ppE173–E187, 1999

Inversion of a generalised Hilbert transform

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(Received 15 February 1999)

Abstract

An integral transform $\mathcal{H}_y$ is defined which reduces to the ordinary Hilbert transform $\mathcal{H}_0$ when $y = 0$, and is useful in some hydrodynamic applications. Although $\mathcal{H}_y$ does not seem to be explicitly invertible for $y \neq 0$ (in contrast to $\mathcal{H}_0^{-1} = -\mathcal{H}_0$), it is readily invertible numerically for y less than a certain precision-dependent bound.

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See http://jamsb.austms.org.au/V40/E013 for this paper and ancillary services, Published May 7 1999, last updated May 7, 1999.

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1 Introduction

The ordinary (doubly-infinite range) Hilbert transform [1] is defined by the Cauchy principal-value integral

$$g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi = \mathcal{H}_0 f(x) \,, \tag{1}$$

where $f(x)$ is a given function defined for all real x. This transform $\mathcal{H}_0$ is explicitly invertible, and indeed its inverse is its negative, i.e.

$$f(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)}{\xi - x} d\xi = -\mathcal{H}_0 g(x) \,, \tag{2}$$

or $\mathcal{H}_0^{-1} = -\mathcal{H}_0$.

Inversion of any integral transform is equivalent to solution of an integral equation of the first kind, and in particular, inversion of Hilbert transforms and some generalisations that retain a Cauchy-singular kernel is the topic of singular integral equation theory [6]. Alternatively, this result can be viewed from a complex function point of view [3], and on the doubly-infinite range is just a simple consequence of Cauchy's theorem, expressing the fact that on the real axis, the real and imaginary parts of a function that is analytic in the upper half plane and vanishes at infinity, are Hilbert transforms of each other.

One of the definitions of a Cauchy principal-value integral is via a limit of the ordinary integral

$$\int_{-\infty}^{\infty} \frac{f(\xi)(\xi - x)}{(\xi - x)^2 + y^2} d\xi \quad \longrightarrow \quad \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x} d\xi \tag{3}$$

as $y \to 0$. It is therefore natural to enquire about the properties of the nonsingular generalised Hilbert transform defined by

$$\mathcal{H}_y f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\ (\xi - x)}{(\xi - x)^2 + y^2} d\xi\,. \tag{4}$$

This is also a natural generalisation in terms of complex functions, since (as a function of both x and y), $\mathcal{H}_y f(x)$ satisfies Laplace's equation for any real input function $f(x)$, and therefore has an interpretation in terms of an analytic continuation from $y = 0$ to $y > 0$. It seems therefore that inversion of $\mathcal{H}_y$ ought to be implied by standard results in complex analysis. Nevertheless, it does not appear straightforward to use either complex analysis or some other natural analytic tools like Fourier transforms, to find a formula for the real-valued inverse of $\mathcal{H}_y$. This inverse does not appear to be stated explicitly in [3] or [5], and it is likely that it must be determined numerically.

A natural partner of the generalised Hilbert transform is the "generalised identity" $\mathcal{I}_y$ such that

$$\mathcal{I}_y f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\, y}{(\xi - x)^2 + y^2} d\xi\,, \tag{5}$$

which is such that $\mathcal{I}_y f \to f$ as $y \to 0_+$, i.e. $\mathcal{I}_y$ becomes the identity in that limit. Then

$$[\mathcal{H}_y + \mathrm{i}\mathcal{I}_y]\, f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - z} d\xi\,, = \mathcal{H}_0 f(z) \tag{6}$$

where $z = x + \mathrm{i}y$. This further indicates a connection to analytic function theory, and to singular integral equations in the complex domain [3]. Integrals equivalent to both $\mathcal{H}_y f(x)$ and $\mathcal{I}_y f(x)$ are introduced by Titchmarsh ([5], p.123) and are used to derive properties of the ordinary Hilbert transform by letting $y \to 0$.

2 Applications

There is an obvious application anywhere that Laplace's equation occurs. In hydrodynamic terms, $\mathcal{H}_y f(x)$ is the velocity potential for a distribution of x-

directed dipoles of strength proportional to $f(x)$ per unit length along the x-axis, and similarly $\mathcal{I}_y f(x)$ is a distribution of y-directed dipoles. Alternatively, $\mathcal{H}_0 f(z) = \mathcal{H}_y f(x) + \mathrm{i}\mathcal{I}_y f(x)$ as defined by equation (6) is the complex velocity induced by a distribution of sources of strength $2f(x)$.

For example, according to thin-airfoil theory [4], a symmetric airfoil is generated in a unit stream of fluid of unit density by a distribution of sources of strength equal to the x-derivative of its thickness. Hence an airfoil of thickness $2f(x)$ yields a pressure field $p(x) = \mathcal{H}_y f'(x)$ at the point (x, y), the small linearised pressure perturbation due to the thin airfoil being proportional to the x-wise velocity perturbation. There are important applications [2] where one seeks to "design" an airfoil's shape $f(x)$ to achieve a certain prescribed pressure distribution $p(x)$. If that pressure is as measured at $y = 0$ (which means effectively on the airfoil itself since it is thin), this design task is simply solved by the explicit inverse Hilbert transform. However, if one seeks to design an airfoil to achieve a given pressure distribution at a fixed offset y, this requires inversion of $\mathcal{H}_y$. Note that in this application, we are more interested in a function whose derivative is the inverse of $\mathcal{H}_y$ than in the inverse function itself, and we shall take that view in the numerical algorithm to follow.

Another application area where this transform occurs is shallow-water ship hydrodynamics [7]. Then, for reasons similar to that in thin airfoil theory, the pressure disturbance caused by a slender ship whose submerged cross-section area is $S(x)$, moving steadily and slowly (strictly at low Froude number) forward in water of constant depth h, is given by $2hp(x) = \mathcal{H}_y S'(x)$. Again, there is no trouble in inverting explicitly from the centreline pressure at $y = 0$ to yield the ship shape function $S(x)$, but there is no equivalent formula for nonzero y.

3 Some Formal Properties

Various identities follow from applying Cauchy's theorem to a function which is analytic in the upper half z-plane, the contour being boundaries of either that whole half plane or the strip between the real axis and the line $y > 0$ constant. Some of these identities can also be derived by elementary real-variable means. In fact, most are special cases of the following, which are obtained by iterating the complex transform defined by (6) with two different values of y. If a and b are any two positive real numbers, then it is not hard to show that for any $f(x)$,

$$[\mathcal{H}_a + \mathrm{i}\mathcal{I}_a]\,[\mathcal{H}_b + \mathrm{i}\mathcal{I}_b]\, f(x) = 2\mathrm{i}\,[\mathcal{H}_{a+b} + \mathrm{i}\mathcal{I}_{a+b}]\, f(x) \tag{7}$$

and

$$[\mathcal{H}_a + \mathrm{i}\mathcal{I}_a]\,[\mathcal{H}_b - \mathrm{i}\mathcal{I}_b]\, f(x) = 0\,. \tag{8}$$

All that is needed is to interchange the order of integration. In (7), the inner integral then has exactly one simple pole in each of the upper and lower half-planes, and the path of integration can be closed in the upper half-plane, giving an integral proportional to the residue at that pole. In (8), both poles are in the lower half-plane, and there is no contribution to the integral.

If we take real and imaginary parts of (7) and (8), the following real operator identities follow:

$$\mathcal{I}_{a+b} = \mathcal{I}_a\mathcal{I}_b = -\mathcal{H}_a\mathcal{H}_b\,, \tag{9}$$

and

$$\mathcal{H}_{a+b} = \mathcal{I}_a \mathcal{H}_b = \mathcal{H}_a \mathcal{I}_b \,. \tag{10}$$

The particular example

$$\begin{aligned}
\mathcal{H}_a \mathcal{H}_b f(x) &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} d\xi\, f(\xi) \int_{-\infty}^{\infty} \frac{(\xi - X)(X - x)\,dX}{[(X-\xi)^2 + a^2]\,[(X-x)^2 + b^2]} \\
&= \frac{1}{\pi^2} \int_{-\infty}^{\infty} d\xi\, f(\xi) \frac{(-\pi)(a+b)}{(\xi - x)^2 + (a+b)^2} \\
&= -\mathcal{I}_{a+b} f(x)
\end{aligned}$$

has been checked directly using Maple.

Although derived assuming $a, b > 0$, these results also appear to hold with either or both parameters zero (and a Cauchy principal value interpretation). They are however false if a and b take opposite signs. So, for example, if it was legitimate to set $a = y > 0$ and $b = -y < 0$, noting that $\mathcal{H}_y$ is an even function of y, we would obtain the inverse $\mathcal{H}_y^{-1} = -\mathcal{H}_y$, which is incorrect unless $y = 0$.

If $a = b = 0$, (9) just verifies the familiar inversion

$$\mathcal{H}_0 \mathcal{H}_0 = -\mathcal{I}_0$$

for the ordinary Hilbert transform. With $a = y > 0$ and $b = 0$, we connect $\mathcal{H}_y$ and $\mathcal{I}_y$ via $\mathcal{H}_0$, e.g.

$$\mathcal{H}_y \mathcal{H}_0 = \mathcal{H}_0 \mathcal{H}_y = -\mathcal{I}_y \,, \tag{11}$$

and

$$\mathcal{I}_y \mathcal{H}_0 = \mathcal{H}_0 \mathcal{I}_y = \mathcal{H}_y \,. \tag{12}$$

These results can also be obtained by direct use of Cauchy's theorem on suitable contours, and are derived explicitly by Titchmarsh [5, p.124].

With $a = b = y$, we relate repeated application of $\mathcal{H}_y$ or $\mathcal{I}_y$ to a single application of $\mathcal{I}_{2y}$, i.e.

$$\mathcal{H}_y\mathcal{H}_y = -\mathcal{I}_y\mathcal{I}_y = -\mathcal{I}_{2y}\,, \tag{13}$$

and similarly

$$\mathcal{I}_y\mathcal{H}_y = \mathcal{H}_y\mathcal{I}_y = \mathcal{H}_{2y}\,.$$

Unfortunately, none of the above yields an explicit inverse for $\mathcal{H}_y$ although they do connect $\mathcal{H}_y$ with $\mathcal{I}_y$ and hence $\mathcal{H}_y^{-1}$ with $\mathcal{I}_y^{-1}$. Thus we know now that

$$-\mathcal{H}_y^{-1} = \mathcal{H}_0\mathcal{I}_y^{-1} = \mathcal{I}_y^{-1}\mathcal{H}_0 = \mathcal{H}_y\mathcal{I}_{2y}^{-1} = \mathcal{I}_{2y}^{-1}\mathcal{H}_y\,.$$

Note again that since $\mathcal{I}_{2y}$ is not the identity unless $y = 0$, $\mathcal{H}_y^{-1} \neq -\mathcal{H}_y$.

4 A Computational Algorithm

In keeping with the applications of interest, instead of inverting $\mathcal{H}_y$ itself, our aim is to develop algorithms for inversion of $\mathcal{H}_y(d/dx)$. That is, given $g(x) = \mathcal{H}_y f'(x)$, we require to find $f(x)$. An integration by parts reduces this task to solving the integral equation

$$-\frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\frac{\partial}{\partial\xi}\frac{(\xi - x)}{(\xi - x)^2 + y^2}d\xi = g(x)\,. \tag{14}$$

Although in some of the applications, we know that $f(x)$ vanishes identically outside a certain finite interval, we choose to make no use of that fact, and one test of the numerical inversion is how well this feature is reproduced by the output data. However, we do assume in the following that the doubly-infinite range of integration can be truncated numerically to a finite range.

For example, suppose that $x = x_j$, $j = 0, 1, 2, \ldots, N$ is a suitable grid of points, where the range (x_0, x_N) is sufficiently wide to encompass all sensible variations in both $f(x)$ and $g(x)$. Then if we approximate $f(\xi) = f_j$ =constant on the j'th interval $x_{j-1} < \xi < x_j$, the integral equation (14) becomes

$$-\frac{1}{\pi}\sum_{j=1}^{N} f_j \left[\frac{(\xi - x)}{(\xi - x)^2 + y^2}\right]_{\xi = x_{j-1}}^{\xi = x_j} = g(x)\,. \tag{15}$$

Let us now collocate at the midpoint $x = \bar{x}_i = (x_{i-1} + x_i)/2$ of the ith interval. Then we have to solve the set of N linear equations

$$\sum_{j=1}^{N} A_{ij} f_j = b_i \tag{16}$$

in N unknowns f_j, where

$$A_{ij} = \frac{x_j - \bar{x}_i}{(x_j - \bar{x}_i)^2 + y^2} - \frac{x_{j-1} - \bar{x}_i}{(x_{j-1} - \bar{x}_i)^2 + y^2}\,, \tag{17}$$

and $b_i = -\pi g(\bar{x}_i)$.

This is a meaningful discretisation for any grid spacings. Let us however specialise to a uniform grid $x_j = x_0 + j\Delta x$. Then

$$A_{ij} = \frac{j-i+\frac{1}{2}}{(j-i+\frac{1}{2})^2+Y^2} - \frac{j-i-\frac{1}{2}}{(j-i-\frac{1}{2})^2+Y^2}\,, \tag{18}$$

where $Y = y/\Delta x$. This is a symmetric matrix with positive constant diagonal elements $A_{ii} = 4/(1+4Y^2)$, and is very easy and economical to set up and invert. The special case $Y = 0$ gives $A_{ij} = 4/(1-4(j-i)^2)$, and this corresponds to inversion of the ordinary Hilbert transform. At $Y = 0$ all off-diagonal elements are negative, and this property also holds for all $Y^2 < 3/4$. For general Y, there is a band of positive elements near the diagonal, but the far off-diagonal elements approach zero through negative values.

A useful test case is $f(x) = (1-x^4)^2$ for $|x| < 1$ and $f(x) = 0$ otherwise. Then the exact transform of $f'(x)$ is

$$g(x) = -\frac{8}{\pi}\Re\left[z^3(1-z^4)\log\frac{z-1}{z+1} + \frac{8}{21} - \frac{8}{5}z^2 + \frac{2}{3}z^4 + 2z^6\right]. \tag{19}$$

This complex form follows from (6), and has also been checked against the 20 lines of output provided by Maple for the real-valued integral from (4).

The inversion algorithm can now be tested by checking whether the original function $f(x)$ is well reproduced by inputting values of the vector b_i computed from this $g(x)$. This must be done using an interval greater than $(-1, 1)$, since $g(x)$ is nonzero both inside and outside this interval and indeed tends to zero

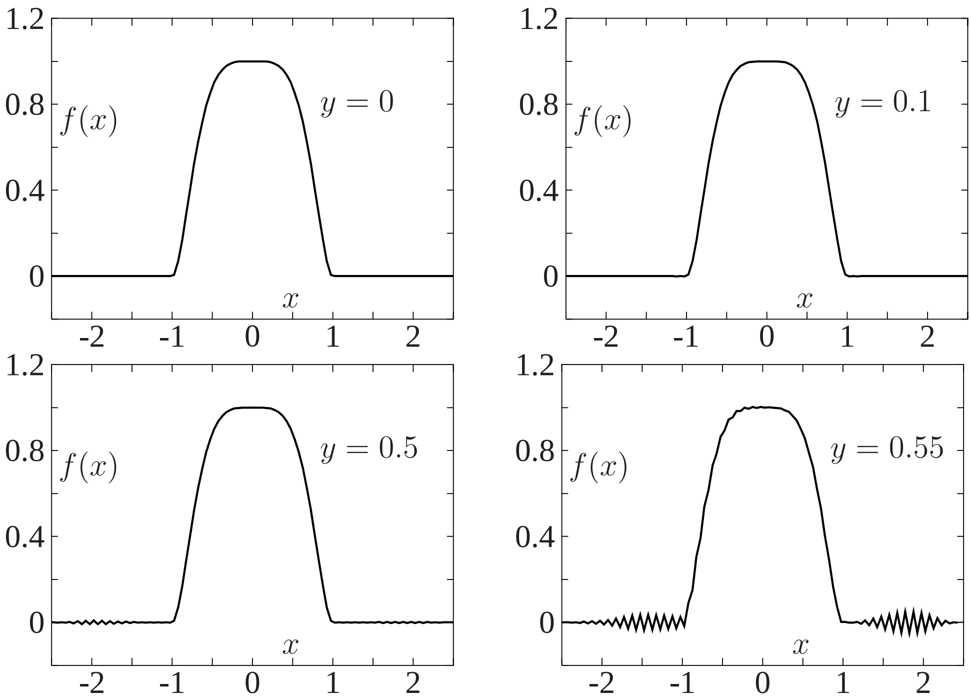

FIGURE 1: Inversion for various y of the exact transform $\mathcal{H}_y f'(x)$, testing numerical ability to recover the original function, namely $f(x) = (1 - x^4)^2$ for $|x| < 1$ and zero otherwise.

through positive values like $1/x^2$ as $|x| \to \infty$. However, the accuracy of inversion seems to be essentially independent of the range (x_0, x_N), so long as it comfortably includes $(-1, 1)$; there is no apparent gain from attempting to capture details of the far field of $g(x)$.

Figure 1 shows results with $(x_0, x_N) = (-2.5, 2.5)$ and $N = 100$ (i.e. $\Delta x = 0.05$) at various y values. The curve for $y = 0$ is graphically indistinguishable from the input $f(x)$, and the output data takes an everywhere-positive value of less than 0.001 in $|x| > 1$ (where $f(x)$ is zero), decaying as $|x|$ increases. Similarly, the curve for $y = 0.1$ is graphically indistinguishable from $f(x)$, with an error less than 0.001, but this error is now oscillatory on a grid scale in $|x| > 1$, the oscillations decaying in amplitude as $|x|$ increases. This pattern holds for all y up to about 0.45, after which the grid-scale oscillations suddenly become more noticeable, e.g. as seen especially in $x < 1$ for the case $y = 0.5$. Eventually these errors become unacceptable, with a clearly random character (e.g. non-symmetric in x), for $y = 0.55$ or greater. Figure 2 shows on a logarithmic scale the mean absolute error as a function of y.

The present (double-precision) algorithm thus seems to be particularly accurate and stable for grid-scaled Y values between about 1 and 10, i.e. for y between Δx and $10\Delta x$, where better than 3-figure mean accuracy is achieved in the present example with $\Delta x = 0.05$. It is also similarly accurate for ordinary Hilbert transforms with $y = 0$ exactly, and although there is some loss of accuracy for small but nonzero y, the results for $0 < Y < 1$ are still acceptable (2 figures instead of 3 in this example). At any fixed y the error decreases rapidly with Δx so long as $y < 10\Delta x$.

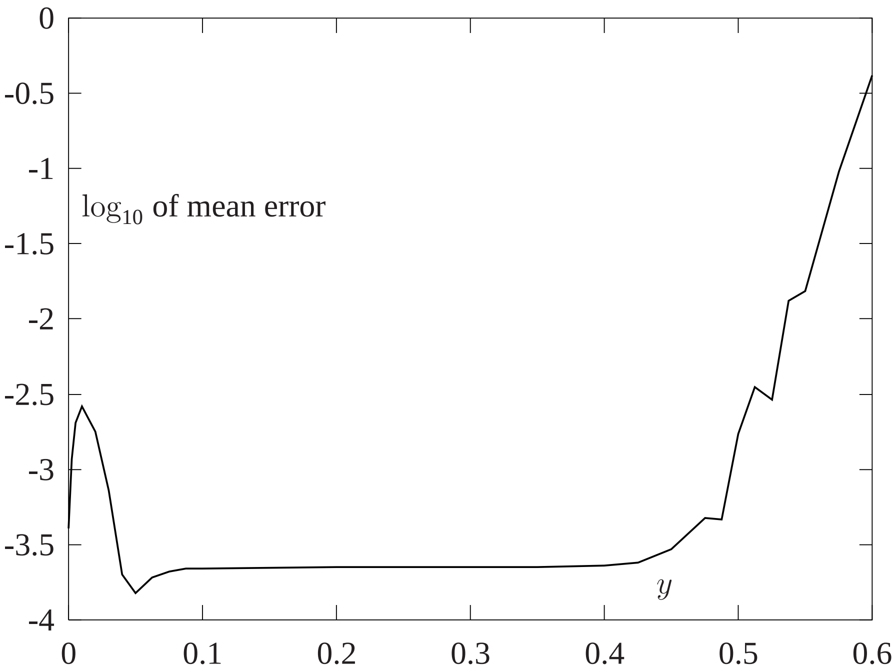

FIGURE 2: Mean error in inversion, as a function of y.

On the other hand, around $y \approx 10\Delta x$, there is a sudden instability-like loss of accuracy in the form of rapidly increasing grid-scale oscillations, and the algorithm then fails altogether for larger y values beyond $12\Delta x$. This eventual failure as y increases is not surprising and probably inevitable, since the matrix A_{ij} loses conditioning as the kernel of the transform it approximates loses singularity. There is thus a real limitation on the achievable accuracy, since at any fixed y, instability will eventually occur as we try to increase accuracy by decreasing Δx. The point at which this instability occurs is sensitive to round-off errors, and in particular occurs at much lower values of $y \approx 3\Delta x$ if the computations are performed in single-precision arithmetic. Meanwhile, however, the range of y over which the method works well in double precision seems to be quite useful.

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