ANZIAM J. 44 (E) ppE82–E113, 2002

Formulation of a tactical logistics decision analysis problem using an optimal control approach

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(Received 6 December 2001, revised 20 June 2002)

Abstract

In recent years, engineers, economists, and military commanders amongst others have placed increasing emphasis on decision making under conditions of uncertainty. Much of life involves making choices under uncertainty, that is, choosing from some set of alternative courses of action in situations where we are uncertain about the actual consequences that will occur for each course of action being considered. It is the field of Decision Analysis that is concerned with the making of rational, consistent decisions, notably under conditions of uncertainty. That is, Decision Analysis helps the decision maker to analyse a complex situation with

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[0]See http://anziamj.austms.org.au/V44/E050 for this article, Published September 6, 2002. ISSN 1446-8735

many different alternatives, states and consequences and to choose the best alternative in light of the information available. The objective of Decision Analysis is to choose a course of action consistent with the basic preferences and knowledge of the decision maker. In this paper we investigate the problem of decision making for the direction of resources within a network of support. This network seeks to mimic how logistic support might be delivered in a military area of operations. It is shown that transitions in the state variables depend upon the status of the network at the end of the previous cycle, the physical distribution decisions taken in the current and previous cycle and the demand for support experienced in the current cycle. By using control theory we are able to formulate the above problem as an optimal control problem, that is, the state variables $X(t)$ are governed by a certain transition function F, and we are seeking the decision stream (optimal controller) for physical distribution actions such that a given Combat Power Cost function is optimised. This latter function is fashioned on some contemporary measures of effectiveness adopted for military logistics. Furthermore, the problem of decision making under uncertainty is also studied by using robust optimal control techniques to formulate the effects of changing situational awareness. A simple case study is given to show the potential of the proposed techniques.

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1 Introduction

Armies worldwide are devoting much time to the consideration of future warfare, what technologies might be involved, what doctrine might be employed, and generally what the nature of warfare might be in the timeframe of 2020 and beyond. These considerations concern not only how armed forces might undertake operations but also how they might support them. It is this latter element that is the concern of logistics.

Contemporary military thought suggests that, in comparison to current methods, logistic support for this future warfare will become more networked in nature and that management of dynamic system behaviour will be needed in order to maximise performance. Based on its application in other problem domains, our proposition is that in this more dynamic and networked environment optimal control would have some beneficial application.

In researching this topic we could not locate any examples of optimal control applied to decision support of logistic networks or general military logistic operations. Therefore, our principal motivation in this paper is to demonstrate and promote the use of optimal control to formulate and investigate the performance of military logistic networks. This is in anticipation that it could play a part in future logistic network command and control systems. In investigating this particular application of optimal control we also wish to include some contemporary performance measures and what we present here, we believe, is a small but useful start to this line of inquiry.

By way of preview the first part of this paper provides a brief discussion of the general topic of decision analysis and some of the available mathematical techniques. We then address the particular problem domain of military logistics and the application of optimal control.

It is important to mention that the optimal control approach we used to study the problem of logistic support decision making in this paper has the advantages of transforming a large network problem to a standard optimal control problem described by a state-space model, which can then be solved using some existing techniques and results. Also, to analyse the time-delay effect and modeling uncertainty in such a state-space model is much easier, and arguably more complete, with our adopted approach in comparison to other Operations Research techniques that might be employed.

2 Decision analysis

2.1 The nature of decision analysis

Operations Research seeks the determination of the best (optimum) course of action of a decision problem under the restriction of limited resources. Although mathematics and mathematical models represent a cornerstone of Operations Research, there is more to problem solving than the construction and solution of mathematical models. Specifically, decision problems usually include important intangible factors that cannot be translated directly in terms of the mathematical model. Foremost amongst these factors is the presence of the human element in almost every decision environment. Indeed, decision situations have been reported where the effect of human behaviour has so influenced the decision problem that the solution obtained from the mathematical model is deemed impractical. Therefore, a decision model is merely a vehicle for summarising a decision problem in a manner that allows systematic identification and evaluation of all decision alternatives of the problem. The decision model can hopefully provide insight and sharpen the intuition of the decision maker, who, in making a decision, must also account for the intangible factors outside the model.

All decision problems have certain general characteristics. These characteristics constitute the formal description of the problems and provide the structure for generating solutions. In this regard, decision problems may be represented by a model in terms of the following elements:

The Decision Maker is responsible for making the decision.

Alternative Courses of Action. An important part of the decision maker's task, over which the decision maker has control, is the specification and description of the alternatives. Given that the alternatives are specified, the decision involves a choice among the alternative course of action.

Events are the scenario or states of the environment not under the control of the decision maker that may occur. Under conditions of uncertainty, the decision maker, does not know for certain which event will occur.

Consequences (also called payoffs, outcomes, benefits or losses), which must be assessed by the decision maker, are measures of the net benefit, or payoff, received by the decision maker. The consequences that result from a decision depend not only on the decision, but also on the event that occurs. Thus there is a consequence associated with each action-event pair.

2.2 Available approaches for decision analysis

There are many powerful tools available to help decision analysis, for example, decisions based on prior information, expected value of perfect information, decision trees, multicriteria decision analysis, sequential decisions, game theory, Bayesian networks, etc, see [18, 19, 15, 16, 17, 30, 12] and the references therein. Since the 1990s, Bayesian networks and decision graphs have attracted a great deal of attention as frameworks for building normative systems. Baysian networks provide formalism for reasoning about partial beliefs under the condition of uncertainty [12]. They have the following features:

- all entities concerned are represented as random variables;
- a graphical structure describes the dependence relations between entities; and

- conditional probability distributions specifies our belief about the strengths of the relations.

For a given number of observations, the probability of different events or hypotheses are computed to help the decision analysis.

Decision trees are concerned with multistage decision processes in which dependent decisions are made in tandem, that is, future decisions will depend on the decision taken now. Multicriteria decision analysis approaches seek to take explicit account of multi (often conflicting) criteria in aiding decision making with the aim of helping decision makers to identify a preferred course of action. This is achieved through a process that identifies and structures the criteria that characterise the available options and then uses this framework to evaluate the options. By using a multiattribute value function approach, Pratt and Belton [17] studied the problem of the architectural options for a command, control, communications and intelligence system.

2.3 Optimal control

On the other hand, undertaking a serious study of a specific dynamic system is often a motivation to improve system behaviour. When this motivation surfaces in explicit form, the subject of optimal control provides a natural framework for problem definition. In the general structure of an optimal control problem, there is a given dynamic system (linear or nonlinear, discrete-time or continuous-time) for which input functions can be specified. There is also an objective function whose value is determined by system behaviour, and is in some sense a measure of the quality of that behaviour. The optimal control problem is that of selecting the input function so as to optimize (maximize or minimize) the objective function.

Optimal control theory has a long history, and research in this area is still quite active now. For some representative work from past to now, see [2, 29, 1, 4, 5, 25, 28, 26, 27, 13, 3, 9, 20, 21, 22, 24, 23] and the references therein. In particular, an integral maximum principle is developed in [29] for a class of nonlinear systems containing time delays in state and control variables. Recently, using optimal control approach to solve manpower planning problem has been studied by the work of [13]. More recently, a number of researchers have investigated the potential of applying control theory to military operational analysis [11, 10, 6, 7]. Cruz and Simaan considered multiagent optimization problems in which one agent is a leader and the others are followers. The leader is that agent who can declare his choice of control first. The concept of a control structure with incentives is explored in which the leader seeks to induce the followers to choose their control vectors in such a way that the leader's objective function is globally optimized. Again, Cruz et al. [7] presented a nonlinear state space mathematical model for a class of dynamical systems that can serve as the basis for a simulation test bed for the investigation of enterprise control. While these works provide military applications of optimal control theory, none of them specifically deal with the area of military logistics.

In this paper, the problem of decision making concerning the direction of resources within a network of military logistic support is studied. Our formulation attempts to mimic key behavioural characteristics of a logistics network that might be employed to support future warfare. By using control theory we demonstrate how to formulate the optimal control problem and apply known results for its analysis.

3 Optimal control and military logistics

3.1 Problem concept

The network nature of future military logistics sees support affected through a system of linked support hubs or nodes. Figure 1 schematically displays an example network topology of nodes and links. The nodes on the directed graph include support bases, distribution hubs, exchange points and customer locations. The links are the multimodal lines of communication or supply routes that connect the nodes and along which logistic support moves.

Armed forces are pursuing these types of network topologies as a means of supporting emergent warfare concepts and for deriving efficiency and effectiveness gains. The provision of logistic support based on a network enables the transfer of resources throughout and across the system, enhancing the responsiveness and robustness of support and reducing layering, linearity and the need for excess redundancy. Historically, support networks would have been more linear in nature with less interconnection between nodes and greater resources located at each node. As a result these older types of networks called on the employment of significantly more resources overall and are thought to be incompatible with future warfare concepts.

We address the problem of how to control a land based distribution system (along the lines shown in Figure 1) to support military operations over time and to meet specific performance criteria. We describe an optimal control model to design appropriate policy constructs for the direction of resources in the network. In terms of the decision problem elements described earlier in the paper we have:

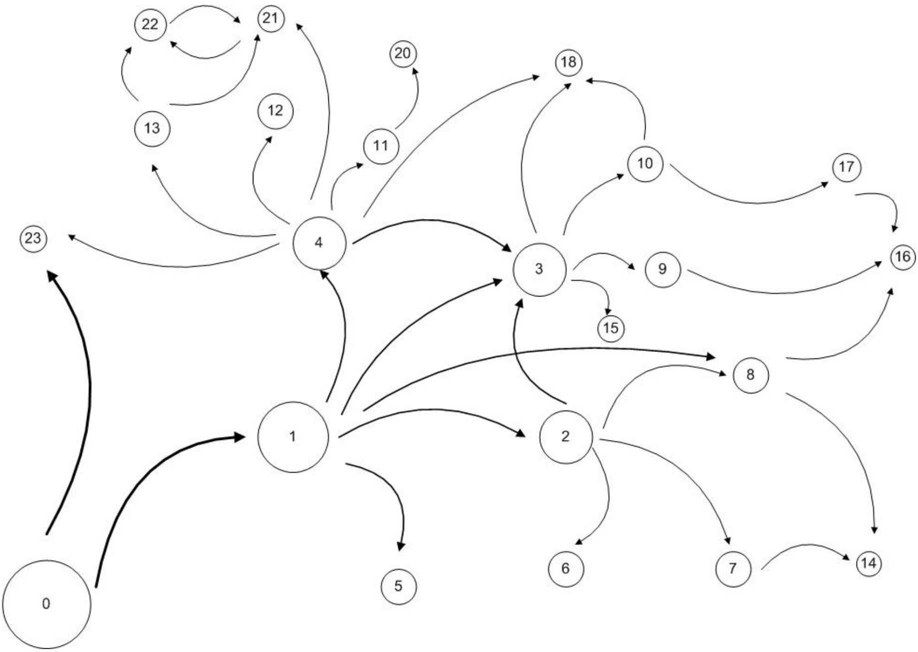

Figure 1: Logistic Network

The Decision Maker. the Logistic Commander who (aided by a command and control system) is responsible for the direction of logistic support throughout the network.

Alternative Courses of Action. the Logistic Commander must make decisions at each phase of operations concerning the movement and location of logistic resources (in our case inventory or physical stock).

Events. the demand for logistic resources at each node in the network, which is largely driven by the employment of combat forces, must be taken into consideration when determining the appropriate course of action.

Consequences. the decisions made by the Logistic Commander will determine the amount of effort to be expended in moving resources around the network and the level of resources required in the network. In terms of our optimal control problem the network performance is fashioned on meeting the demand for logistic support while optimising the following criteria:

Physical Distribution effort. we seek to minimise the combat power required to maintain and protect the distribution of material throughout the network. Maintaining safe lines of communication or supply routes throughout the network can potentially divert significant combat and combat support resources away from other operational aims.

Footprint. we seek to minimise the amount of material located throughout the network. The term footprint has been loosely used to describe the presence projected by support elements in an Area of Operations and is predominantly thought of in terms of personnel, equipment and supplies deployed. Reference to 'small' or 'large' logistic footprint is often made in military doctrinal thinking. All things being equal, smaller footprints are most desired for combat effectiveness because, for example, they reduce the potential for enemy detection and disruption, they reduce the need for force protection, they enhance the mobility of a force, and provide for nimble operational options. We use footprint to represent another opportunity cost to combat power of having to protect logistic resources located in the network.

3.2 General formulation of a logistic decision making problem

The logistic commander makes decisions at each phase in keeping with the overall mission intent and operational aims. Therefore, at each phase, the decision set $Y(t)$ consists of a number of components

$$Y(t) = \{Y_1(t), Y_2(t), \dots, Y_L(t)\} ,$$

where t stands for the phase and $Y_i(t)$ is the decision taken in relation to physical distribution along the ith supply route of the network.

Each of the decisions is taken from among a discrete set of possible choices described above. The sequence of decisions taken by the logistic commander over a time horizon is referred to as the decision stream. We denote

$$Y = \{Y(0), Y(1), \dots, Y(T-1)\} ,$$

as the decision stream for a T-phase operation. Clearly, the number of possible decision streams for even a simple operation can easily become unmanageably large (owing to the number of options available at each cycle), and thus we are burdened by the curse of dimensionality.

The consequences of the logistic commander's decisions at each time phase can be measured in several ways. Let

$$X(t) = [x_1(t), x_2(t), \dots, x_N(t)] \ ,$$

be the vector of logistic resources at the N network nodes at operational phase t, then the decision stream can be expressed as a multistage decision process, incorporating time-delay, described by a state-space model:

$$X(t+1) = F\,[t, X(t), Y(t), Y(t-1), W(t)] \ , \tag{1}$$

where $F[\cdot]$ is a transition function. $W(t)$, in mathematical terms, represents a random disturbance for the system. At each phase, that is, $F[i, X(i), Y(i), Y(i-1)] \to X(i+1)$, the logistic commander wishes to select $Y(t)$ so that a prescribed performance function

$$P = \sum_{t=0}^{T-1} H[t, X(t), Y(t)] + g[X(T)] \tag{2}$$

is optimised (minimised or maximized), where $X(0)$ is the initial logistic resources positioned at each location at the beginning of the time horizon, and $X(T)$ is this same property at the end.

From (1) and (2), we can see that finding the optimal decision stream Y is then an optimal control problem. The decision variables, $Y(t)$, at each phase are control variables and the $X(t)$ are the state variables.

3.3 A tactical logistic decision analysis problem

Now, we formulate the tactical logistics decision analysis problem in a mathematical model. For presentation convenience, we introduce the following notation: $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$), means that $X - Y$ is positive semi-definite (respectively, positive definite). $\|\cdot\|$ denotes the Euclidean vector norm.

Indices:

j: indices for location nodes, $j \in \mathcal{N} = \{0, 1, \ldots, N\}$

k: indices for supply routes, $k \in \mathcal{L} = \{1, \ldots, L\}$

t: time on time horizon $[0, T]$

Sets:

- $S_j = \{k \in \mathcal{L}$ such that logistics resources can be supplied along route k *to* node $j\}$;
- $C_j = \{k \in \mathcal{L}$ such that logistics resources can be supplied along route k *from* node $j\}$.

Variables:

- $x_j(t)$ = Stock of logistic resources at location j at the beginning of time period t ;
- $y_k(t)$ = Stock dispatched for supply along route k during time period t ;

- $w_j(t)$ = Demand for stock at location node j during time period t .

In a balanced situation the physical stock of logistic resources at location j, $j = 1, 2, \ldots, N$ at the commencement of time period $t+1$ is

equal to The physical stock of logistic resources at location j at the beginning of time period t: $[x_j(t)]$

Plus The amount of stock received at location j from along route k during time period t, where $k \in S_j$ for all k: $[y_k(t-1)]$ (In our model we assume there is a delay of one time period between the dispatch of material from a supply location and receipt at a receiving location.)

Less The amount of stock dispatched for supply from location j along route k during time period t, where $k \in C_j$, for all k: $[y_k(t)]$

Less The local demand for stock at location j during time period t: $[w_j(t)]$.

So, we have

$$\begin{aligned}
&x_j(t+1) = x_j(t) + \sum_{k \in S_j} y_k(t-1) - \sum_{k \in C_j} y_k(t) - w_j(t) \,, \\
&\quad j = 1, 2, \ldots, N \\
&x_0(t) = M \,, \quad y_k(-1) = 0
\end{aligned} \tag{3}$$

where M is some fixed number.[1]

[1] $x_0(t)$ in the mathematical model represents the originating source for stock flowing into the network and therefore a system boundary.

We now consider stock loss from this balanced situation due to disruption from operational threats (enemy or neutral activity) or environmental threats (stock contamination etc). We consider the affect of this in two parts of the network: at location nodes and along supply routes.

Let

- A_j = Proportion of stock at location j that is available for the next time period, $A_j \in [0, 1]$, $j = 1, 2, \ldots, N$;
- B_k = Proportion of stock along the supply route k that can be successfully supplied, $B_k \in [0, 1]$, $k = 1, 2, \ldots, L$.

A_j and B_k reflect the level of support interdiction and general stock losses within the network. We contend that it would be possible, given situational awareness and intelligence concerning threats in the network, to construct discrete probability distributions for each A_j, $j \in \mathcal{N}$ and B_k, $k \in \mathcal{L}$. We use $\bar{A}_j$ and $\bar{B}_k$ to represent the expected values for A_j and B_k respectively.

The transition function then becomes, for $j = 1, 2, \ldots, N$,

$$x_j(t+1) = \bar{A}_j x_j(t) + \sum_{k \in S_j} \bar{B}_k y_k(t-1) - \sum_{k \in C_j} y_k(t) - w_j(t) . \quad (4)$$

More generally

$$X(t+1) = AX(t) + B_0 Y(t) + B_1 Y(t-1) - W(t) ,$$
$$X(0) = X_0 , \quad Y(-1) = 0 , \quad (5)$$
$$X_{\min} \le X(t) \le X_{\max} , \quad Y_{\min} \le Y(t) \le Y_{\max} , \quad (6)$$

where the bounds on the state variable $X(t)$ and the control variable $Y(t)$ reflect the physical limitations in capacity at locations and along supply routes, or in the case of $X_{\min}$ the requirements for reserve stock, and

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}, \quad Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_L(t) \end{bmatrix},$$

$$Y(t-1) = \begin{bmatrix} y_1(t-1) \\ y_2(t-1) \\ \vdots \\ y_L(t-1) \end{bmatrix}, \quad W(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ \vdots \\ w_N(t) \end{bmatrix},$$

$$A = \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A}_N \end{bmatrix}$$

$$B_0 = (B_0)_{jk} = \begin{cases} -1 & \text{if } k \in C_j \\ 0 & \text{otherwise} \end{cases}$$

$$B_1 = (B_1)_{jk} = \begin{cases} \bar{B}_k & \text{if } k \in S_j \\ 0 & \text{otherwise} \end{cases}$$

With regard to the overall behaviour of the logistic network we are interested in meeting the demand for logistic resources and optimising two performance criteria.

Let

$Q_j =$ Opportunity Cost to combat power of locating and protecting logistics resources at location j, $j = 1, 2, \ldots, N$

$R_k =$ Opportunity Cost to combat power of both maintaining and protecting distribution effort along supply route k, $k = 1, 2, \ldots, L$.

Q_j and R_k are theoretical constructs at this stage. However, as with most distribution processes, it is common place in military distribution networks to assign priorities (or levels of importance) among locations and lines of communications. In this regard, we might expect locations and supply routes that are accorded low priority to present high opportunity costs for combat power. In addition, the opportunity cost of protecting a supply route is likely to be proportional to its length.

Using these parameters we define

$$R = \begin{bmatrix} R_1 & 0 & \cdots & 0 \\ 0 & R_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_N \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Q_L \end{bmatrix}.$$

Then in terms of our criteria previously outlined:

1. Footprint—We wish to minimise the average combat power expended in protecting logistic resources located in the network.

2. Physical Distribution Effort—We wish to minimise the average combat power expended in maintaining and protecting distribution effort along supply routes.

Finally, with these performance criteria in mind we construct a relationship which we loosely define as the Combat Power Cost function.

$$P = \sum_{t=0}^{T-1} \Big(X^T(t) Q X(t) + Y^T(t) R Y(t) \Big) + X^T(T) Q X(T) \,. \qquad (7)$$

We are not aware of the exact form of the relationships between $X(t)$, $Y(t)$ and the opportunity cost coefficients Q and R respectively but we postulate that they will not be linear. Smaller amounts of stock positioned at locations and distributed along supply routes may have combat power protection requirements met relatively easily, however, as these stock levels increase the protection requirements grow more quickly. For our purposes we have assumed a quadratic relationship.

4 Problem solution—optimal control approach

In this section, we solve the problem formulated in the previous section by employing standard optimal control techniques (see for example [2, 13]). To this end, recall some optimal control results.

Consider the following linear discrete-time system

$$\begin{aligned} & x(t+1) = Ax(t) + B_0 u(t) + B_1 u(t-1)\,, \\ & x(0) = x_0\,, \quad u(-1) = 0 \\ & x_{\min} \le x(t) \le x_{\max}\,, \quad u_{\min} \le u(t) \le u_{\max} \end{aligned} \tag{8}$$

where $x(t)$ is a n-dimensional state vector, $u(t)$ is a r-dimensional control input vector, A, B_0 and B_1 are $n \times n$ and $n \times r$ known constant matrices.

The system cost function is assumed to be quadratic:

$$J_T = \frac{1}{2} x^T(T) Q x(T) + \sum_{t=0}^{T-1} \frac{1}{2} \left\{ x^T(t) Q x(t) + u^T(t) R u(t) \right\} \tag{9}$$

where the weighting matrices $Q(t)$ and $R(t)$ in (9) are positive definite.

The optimal control problem is to *find a sequence of control vectors* $u(0)$, $u(1)$, ..., $u(T-1)$ *such that (9) is minimised while (8) are satisfied.*

From the results in [14], we know that the optimal solution for (8) and (9) is

$$\begin{aligned} x(t) &= \mathrm{Sat}_x\left\{-Q^{-1}\Big[-\lambda(t)+A^T\lambda(t+1)\Big]\right\}, \quad t=1,2,\ldots, \qquad (10)\\ u(t) &= \mathrm{Sat}_u\left\{-R^{-1}\Big[B_0^T\lambda(t+1)+B_1^T\lambda(t+2)\Big]\right\}, \\ &\qquad t=1,\ldots,T-1 \qquad (11)\\ x(T) &= \mathrm{Sat}_x\{Q^{-1}\lambda(T)\} \qquad (12)\end{aligned}$$

where

$$\mathrm{Sat}_x(v_i)=\begin{cases} x_{\max,i} & \text{if } v_i>x_{\max,i}\\ v_i & \text{if } x_{\min,i}\le v_i\le x_{\max,i}\\ x_{\min,i} & \text{if } v_i<x_{\min,i}\end{cases}$$

$$\mathrm{Sat}_u(\sigma_j)=\begin{cases} u_{\max,j} & \text{if } \sigma_j>u_{\max,j}\\ \sigma_j & \text{if } u_{\min,j}\le \sigma_j\le u_{\max,j}\\ u_{\min,j} & \text{if } \sigma_j<u_{\min,j}\end{cases}$$

and the indices i and j represent the ith and jth elements of state x_i, $i=1,2,\ldots,n$ and control u_j, $j=1,2,\ldots,r$ respectively. While $\lambda(t)$, $t=0,1,\ldots,T$ is a vector of Lagrange multipliers at time t, and $\lambda(T+1),\ldots$ are defined as zero vectors.

To locate this optimal solution we employ the following time-delay control design algorithm:

1. solve (10–12) for a fixed set of Lagrange multiplier vector $\lambda(t)$, $t=0,1,\ldots,T$;

2. the value of $\lambda(t)$ is improved through a gradient type iteration

$$\lambda^{i+1}(t) = \lambda^i(t) + \delta^i d^i(t)\,, \tag{13}$$
$$d^i(t) = Ax(t) + B_0 u(t) + B_1 u(t-1) - x(t+1)\,, \tag{14}$$

where δ^i is a small number.

Remark: Indeed, we can remove the delay term $u(t-1)$ by creating an extra state variable, that is, let $z(t+1) = u(t)$ and $z(0) = 0$, then we have from (8)

$$\begin{aligned} &x(t+1) = Ax(t) + B_0 z(t+1) + B_1 z(t)\,, \\ &x(0) = x_0\,, \quad z(0) = 0\,, \\ &x_{\min} \le x(t) \le x_{\max}\,, \quad u_{\min} \le z(t) \le u_{\max}\,. \end{aligned} \tag{15}$$

Define the Hamiltonian of system (15) with cost function (9) as

$$H(t) = \frac{1}{2}\left\{x^T(t)Qx(t) + u^T(t)Ru(t)\right\} + \lambda^T(t+1)f(t)\,,$$

where

$$f(t) = Ax(t) + B_0 z(t+1) + B_1 z(t)\,.$$

By some standard manipulation on the Hamiltonian $H(t)$, we have the following

$$\lambda(T+1) = 0\,, \tag{16}$$
$$\lambda(T) = Qx(T)\,, \tag{17}$$
$$\lambda(t) = Qx(t) + A^T\lambda(t+1)\,, \tag{18}$$
$$Ru(t) = B_0^T\lambda(t+1) + B_1^T\lambda(t+2)\,. \tag{19}$$

See that the optimal solution for system (8) driven from (16–19) is identical to that from (10–12). However, since (16–19) is derived by the 'delay-free' system (15), this can be conveniently computed by the software DMISER3 (see [8] for details).

5 Robust decision making modelling with uncertainties

In this section we consider uncertainties that might exist in the logistic network and in our mathematical representation of it. Our motivation is based on the fact that, in practice, it is almost impossible to get an exact mathematical model of a dynamical system due to the complexity of the system, the difficulty of measuring various parameters, environmental noises, uncertain and/or time-varying parameters, etc. Indeed, the model of a system to be controlled must consider the affects of possible perturbations in order to design a robust controller.

In the general case, the perturbed linear stochastic system of (8) is assumed to have the following form:

$$\begin{aligned} x(t+1) &= [A+\Delta A]\,x(t) + B_0 u(t) + [B_1+\Delta B_1]\,u(t-1) \\ x(0) &= x_0\,, \quad u(-1)=0 \\ x_{\min} &\le x(t) \le x_{\max}\,, \quad u_{\min} \le u(t) \le u_{\max}\,, \end{aligned} \tag{20}$$

where all the variables are as in (8), except the parameter uncertainties ΔA and ΔB_1 assumed to be bounded, that is,

$$\|\Delta A\| \le a\,, \quad \|\Delta B_1\| \le b_1\,, \tag{21}$$

where a and b_1 are known positive numbers.

For the case of robust analysis on the problem formulated in Section 3, we wish to investigate the effects of situational awareness. Recall that in our logistic network decision problem A_j and B_k are parameters measured with reference to our level of situational awareness of operational and environmental threats across the network. Our assumption is that for cases of high situational awareness we are capable of accurately estimating $\bar{A}_j$ and $\bar{B}_k$; and by extension the possible ranges of A and B_1 are quite small. For cases where we are less situationally aware, this is not so and possible ranges of A and B_1 are much greater.

Hence, for robust analysis, the nominal model (5) should have the following form

$$\begin{aligned} X(t+1) &= [A+\Delta A]X(t) + B_0Y(t) \\ &\quad + [B_1+\Delta B_1]Y(t-1) - W(t)\,, \quad X(0) = X_0\,, \qquad (22) \\ X_{\min} &\le X(t) \le X_{\max}\,, \quad Y_{\min} \le Y(t) \le Y_{\max}\,. \qquad (23) \end{aligned}$$

We wish to design an optimal control $Y(t)$ (choose the best physical distribution alternative) such that the Combat Power Cost Function (7) is minimised subject to (22–23). From Robust Optimal Control Theory, we know that if we can solve this problem with the upper bounds of the uncertainties (21) then the optimal solution can also be applied to the situation when the uncertainties have a smaller bound. Furthermore, if we cannot solve the above logistic network control problem with the uncertainties described in (21) then it is suggested that network redesign may be needed, for example in terms of the capacity limitations described at (23).

6 Example

In this section, we use a numerical example to illustrate the modelling concept and the solution approach to the logistic network support decision making problem. The logistic network topology for the example is shown in Figure 2. Applying the system (5–6) and Combat Power Cost Function (7) to this example network for

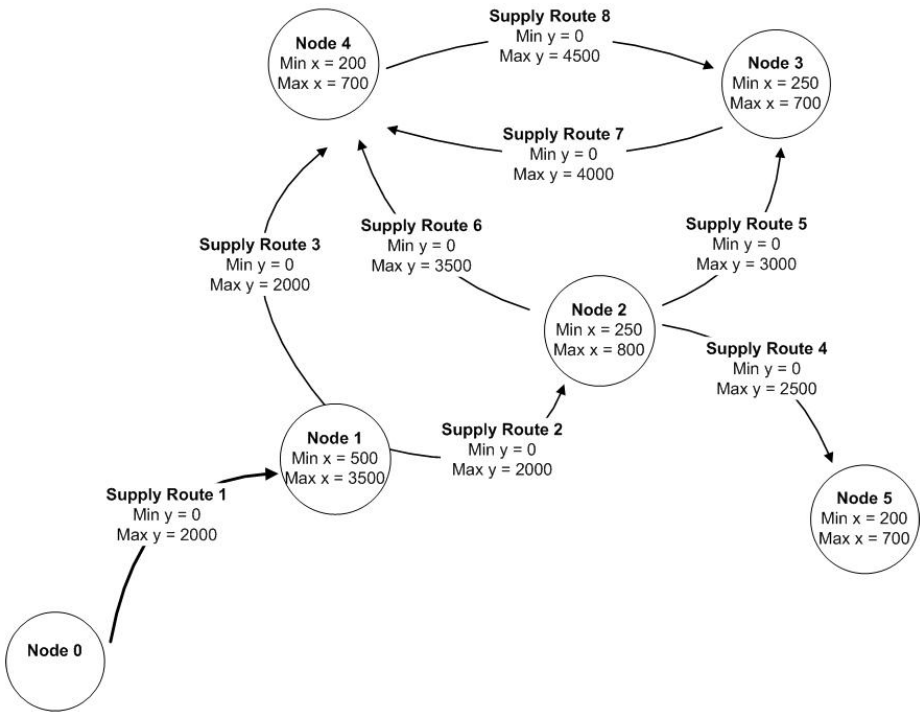

Figure 2: Example of Logistic Network

a time horizon consisting of five phases ($T = 5$), we set

$$A = \begin{bmatrix} 0.95 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0 & 0.85 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.95 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.87 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.75 & 0 & 0 & 0.7 \\ 0 & 0 & 0.8 & 0 & 0 & 0.8 & 0.7 & 0 \\ 0 & 0 & 0 & 0.85 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 3500 \\ 800 \\ 400 \\ 400 \\ 200 \end{bmatrix}, \quad W(0) = \begin{bmatrix} 1000 \\ 150 \\ 80 \\ 100 \\ 70 \end{bmatrix}, \quad W(1) = \begin{bmatrix} 750 \\ 200 \\ 250 \\ 150 \\ 50 \end{bmatrix},$$

$$W(2) = \begin{bmatrix} 700 \\ 300 \\ 200 \\ 75 \\ 100 \end{bmatrix}, \quad W(3) = W(4) = W(5) = \begin{bmatrix} 600 \\ 300 \\ 200 \\ 75 \\ 100 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2.5 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

After around 400 iterations of the time-delay control design algorithm described in Section 4 we obtained the minimal opportunity cost described by the Combat Power Cost function. Figure 3 shows an example of the convergence of $d(k)$ within the algorithm. This convergence reflects how iteratively the decisions concerning physical distribution ($Y(t)$) and the resultant location stock levels ($X(t)$) meet the boundary conditions (6), satisfy the local demand (that is, meet the transition function (5)) and provide an optimal cost outcome.

Figure 4 shows the progress of the Combat Power Cost function for each iteration of the algorithm. In the early iterations the state and control variables meet the boundary conditions but fail the transition function requirements, these are infeasible solutions, that is, the function values are increasing because the iterates are infeasible. As the algorithm progresses and these variables eventually meet the further constraints of the transition requirements, feasible and optimal physical distribution decisions are derived for the network. This explains the upward trajectory of Figure 4 even though we are addressing a minimisation problem. Also, from our analysis on the numerical simulation example, it is noted that the starting/initial values for the co-state variables have some effect on the overall state trajectories in a short time period, that is, the time to meet the boundary conditions. But, if the final time T is relatively large, then the effect will play a very little role. Also, note

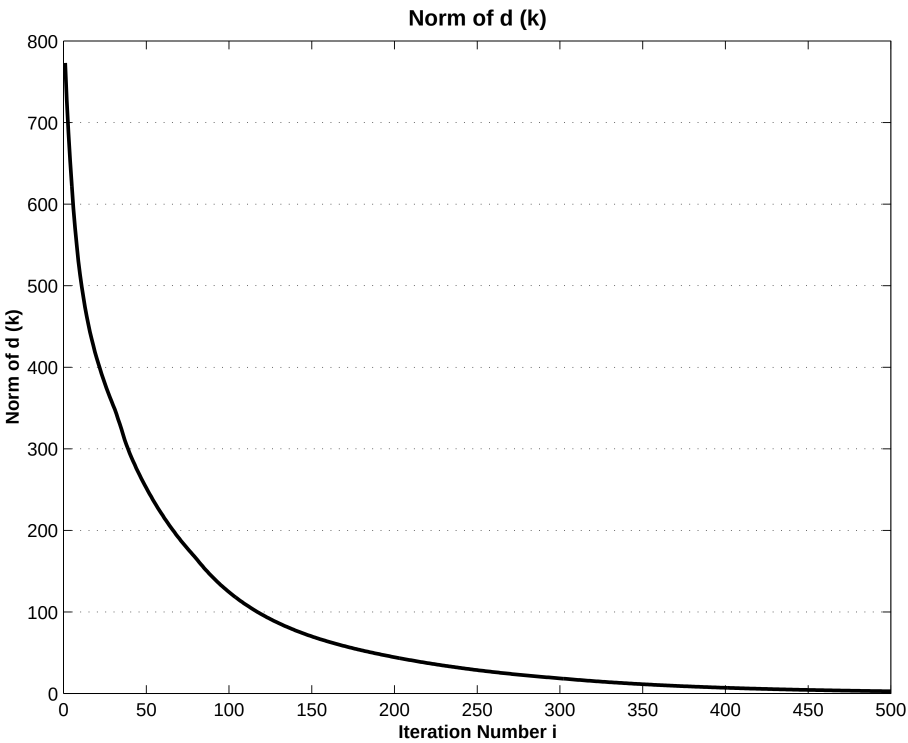

Figure 3: Convergence of $d(k)$

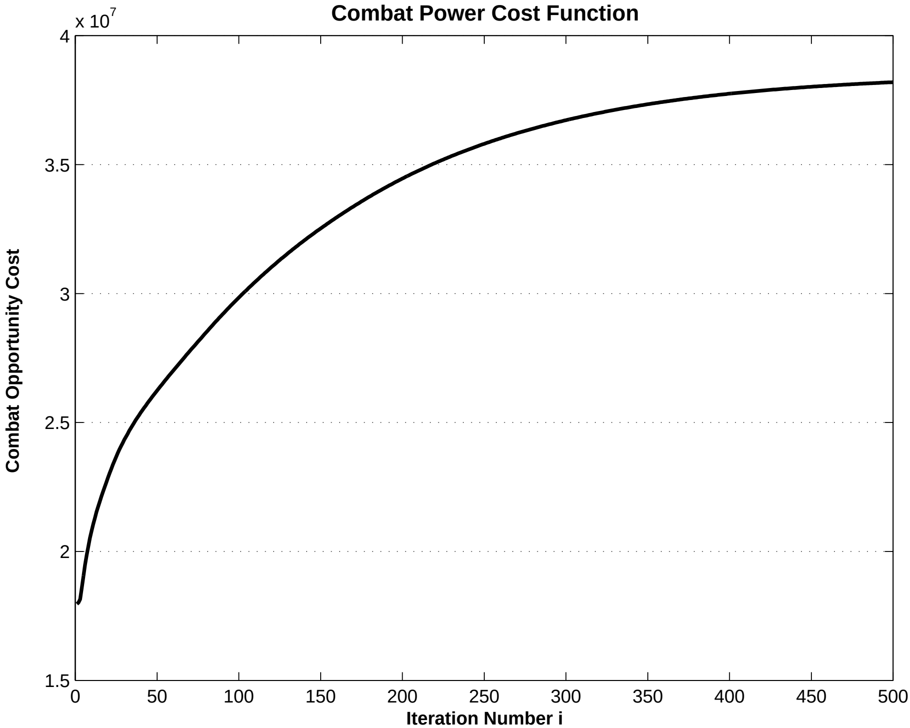

Figure 4: Combat power opportunity cost versus iteration

that a computational procedure for solving combined discrete time optimal control and optimal parameter selection problems subject to general constraints is presented in [8] by converting the underlying problem into a nonlinear programming problem which can be solved using standard optimization software.

7 Conclusion

In this paper we have presented a mathematical model that is aligned with contemporary military thought concerning the operation of future military logistic networks. The general nature of the model is a multistage decision process, incorporating time delays, which we solve using an algorithmic optimal control approach. Furthermore, the problem is formulated for changing levels of situational awareness and a numerical example is provided to demonstrate the feasibility of the time delay control design algorithm. We do believe that this approach has potential and that with further research has application in logistic decision support systems.

Acknowledgment: the authors are grateful to the reviewers and Prof K. L. Teo for the valuable comments and suggestions which have improved the presentation of the paper.

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