An optimal battery interchange policy for an electric car powered by a mobile solar power station

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Abstract

The World Solar Challenge is a 3000 kilometre race from Darwin to Adelaide, across the Australian continent, for solar powered racing cars. Annesley College accompanied the 2001 event in an electric car powered by batteries. While one battery was used to power the car another was charged from a solar panel carried by a mobile solar power station. When the first battery became empty the batteries were interchanged and the first battery put on charge. The process was repeated throughout the event. In this paper we find

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a policy for interchanging the batteries that maximises the distance travelled each day.

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1 Introduction

Annesley College have been racing solar cars in the World Solar Challenge since 1990. After the 1996 race the team decided that, rather than building another solar car, they would develop a commuter car that could be powered from renewable solar energy. They took a small conventional car and replaced the engine, transmission and petrol tank with an electric motor, fixed-ratio reduction drive and a battery.

The car has a range of about 50 km, and would normally be driven on short trips around town and then recharged using renewable energy from the electricity grid. The trip from Darwin to Adelaide required the team to drive 300 km each day and generate their own energy. To do this, the team used a large photovoltaic array on the back of a truck to continually recharge spare batteries. These batteries were periodically interchanged with the battery in the car.

Our problem was to determine a battery interchange policy that would maximise the distance travelled by the car each day.

2 Modelling the problem

We start each day at time $t_0 = 0$, and assume that at this time the battery in the car contains energy E_0 and the second battery is empty. Battery packs will be swapped at times $t_1, t_2, \ldots, t_n$, to be determined. The journey finishes each day at time $t_{n+1} = T$. The aim is to find switching times $0 < t_1 < t_2 < \cdots < t_n < T$ that maximise the distance travelled during the interval $[0, T]$. During each interval (t_{i-1}, t_i) we drive at a constant speed so that the battery in the car becomes empty at time t_i. We then swap batteries and continue.

The power p required to hold a constant speed v is $p = f(v)$ where $f : [0, \infty) \to [0, \infty)$ is

$$f(v) = av + bv^2 + cv^3 \,,$$

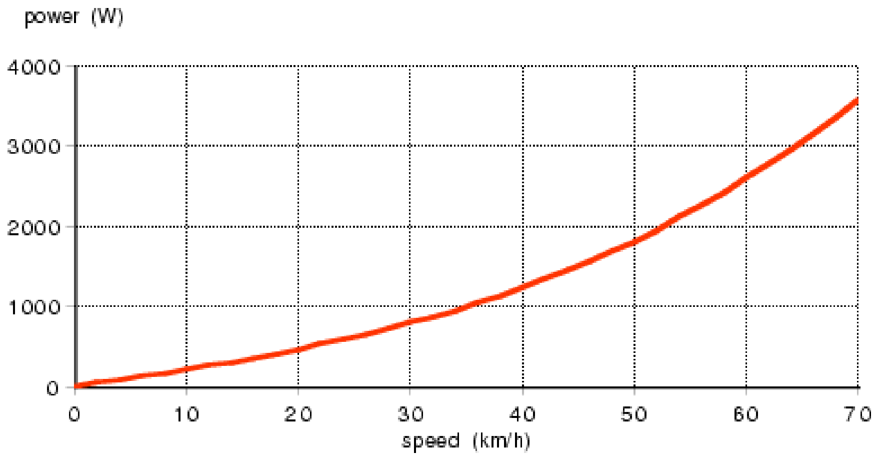

Figure 1: The power required to maintain a constant speed is positive, increasing and convex.

where a, b and c are known positive constants. An example is shown is Figure 1.

The constant speed v that can be maintained with power supply p is $v = \varphi(p)$ where $\varphi : [0,\infty) \mapsto [0,\infty)$ is given by $\varphi(p) = f^{-1}(p)$. Because $f(v)$ is positive, increasing and convex with

$$f(0) = 0 \quad \text{and} \quad f'(0) = a\,,$$

it is clear that $\varphi(p)$ is positive, increasing and concave with

$$\varphi(0) = 0 \quad \text{and} \quad \varphi'(0) = a^{-1}\,.$$

The total energy collected to time t is given by

$$E(t) = E_0 + \int_{[0,t]} s(\tau)\,d\tau$$

where $s(\tau)$ is the known solar power generated by the solar panel at time τ. For convenience we write $E_i = E(t_i)$ and $s_i = s(t_i)$ for all $i = 0, \ldots, n+1$. We also write $\Delta E_{-1} = E_0$ and $\Delta E_i = E_{i+1} - E_i$ and $\Delta t_i = t_{i+1} - t_i$ for each $i = 0, \ldots, n+1$.

The constant battery power used on the interval $[t_i, t_{i+1}]$ depends on the energy collected during the previous interval, and is

$$\bar{p}_i = \frac{\Delta E_{i-1}}{\Delta t_i}\,.$$

The speed of the car on the interval $[t_i, t_{i+1}]$ is

$$\bar{v}_i = \varphi(\bar{p}_i)\,.$$

3 An elementary bound for the total distance

The distance travelled by the electric car on the interval $[0, T]$ is

$$x(t_1, \ldots, t_n) = \sum_{i=0}^{n} \bar{v}_i \Delta t_i = T \sum_{i=0}^{n} \varphi(\bar{p}_i) \frac{\Delta t_i}{T}\,,$$

and from the concavity of $\varphi(p)$ it follows that

$$x(t_1, \ldots, t_n) \leq T\varphi\left(\sum_{i=0}^{n} \bar{p}_i \frac{\Delta t_i}{T}\right) = T\varphi\left(\frac{E_n}{T}\right) < T\varphi\left(\frac{E(T)}{T}\right)$$

for all such subdivisions.

Theorem 1 *The maximum distance X_T travelled in the period $[0, T]$ satisfies the inequality*

$$X_T < T\varphi\left(\frac{E(T)}{T}\right)\,. \tag{1}$$

4 Necessary conditions for a maximal journey

To maximise the distance travelled we first calculate some derivatives. For each $i = 1, \ldots, n-1$ we have

$$\begin{aligned}\frac{\partial}{\partial t_i}&[\varphi(\bar{p}_{i-1})\Delta t_{i-1} + \varphi(\bar{p}_i)\Delta t_i + \varphi(\bar{p}_{i+1})\Delta t_{i+1}]\\ &= \omega(\bar{p}_{i-1}) - \omega(\bar{p}_i) + [\varphi'(\bar{p}_i) - \varphi'(\bar{p}_{i+1})]s_i\,,\end{aligned}$$

and

$$\frac{\partial}{\partial t_n}\varphi(\bar{p}_{n-1})\Delta t_{n-1} + \varphi(\bar{p}_n)\Delta t_n = \omega(\bar{p}_{n-1}) - \omega(\bar{p}_n) + \varphi'(\bar{p}_n)s_n\,,$$

where $\omega(p) = \varphi(p) - \varphi'(p)p$. See that $\omega'(p) = -\varphi''(p)p > 0$ and hence $\omega(p)$ increases with p. Now it follows that for each $i = 1, \ldots, n-1$ the equations

$$\frac{\partial x(t_1, \ldots, t_n)}{\partial t_i} = 0\,,$$

can be rewritten in the form

$$\omega(\bar{p}_{i-1}) - \omega(\bar{p}_i) + [\varphi'(\bar{p}_i) - \varphi'(\bar{p}_{i+1})]s_i = 0\,,$$

and the final equation

$$\frac{\partial x(t_1, \ldots, t_n)}{\partial t_n} = 0\,,$$

gives

$$\omega(\bar{p}_{n-1}) - \omega(\bar{p}_n) + \varphi'(\bar{p}_n)s_n = 0\,.$$

We begin by considering this final equation. Since

$$\varphi'(\bar{p}_n)s_n > 0\,,$$

we deduce that $\omega(\bar{p}_{n-1}) < \omega(\bar{p}_n)$ and, since $\omega(p)$ increases with p, that $\bar{p}_{n-1} < \bar{p}_n$. Since $\varphi'(p)$ decreases when p increases the equation

$$\omega(\bar{p}_{n-2}) - \omega(\bar{p}_{n-1}) + [\varphi'(\bar{p}_{n-1}) - \varphi'(\bar{p}_n)]s_n = 0\,,$$

shows us that $\omega(\bar{p}_{n-2}) < \omega(\bar{p}_{n-1})$ and hence that $\bar{p}_{n-2} < \bar{p}_{n-1}$. By considering each of the other equations in turn we find that $\bar{p}_0 < \bar{p}_1 < \bar{p}_2 < \cdots < \bar{p}_n$.

Theorem 2 *A necessary condition that $x(t_1, \ldots, t_n)$ is maximised is that*

$$\omega(\bar{p}_{i-1}) - \omega(\bar{p}_i) + [\varphi'(\bar{p}_i) - \varphi'(\bar{p}_{i+1})]s_i = 0 \tag{2}$$

for each $i = 1, \ldots, n-1$ and that

$$\omega(\bar{p}_{n-1}) - \omega(\bar{p}_n) + \varphi'(\bar{p}_n)s_n = 0\,. \tag{3}$$

If these conditions are satisfied then the average power used on each subinterval satisfies the inequalities

$$\bar{p}_0 < \bar{p}_1 < \cdots < \bar{p}_n\,. \tag{4}$$

5 The case in which solar power is constant

If $s(t) = S$ for all $t \in [0, T]$ then we can construct a simple numerical procedure to calculate the solution. In this special case we have

$$\bar{p}_0 = \frac{E_0}{\Delta t_0} \quad \text{and} \quad \bar{p}_i = \frac{S\Delta t_{i-1}}{\Delta t_i}$$

for each $i = 1, 2, \ldots, n$. If we nominate a particular value $\bar{p}_n$ for the power on the final interval then we determine $\bar{p}_{n-1}, \ldots, \bar{p}_0$ by first solving the equation

$$\omega(\bar{p}_{n-1}) = \omega(\bar{p}_n) - S\varphi'(\bar{p}_n)\,,$$

and then recursively solving the equations

$$\omega(\bar{p}_{n-k-1}) = \omega(\bar{p}_{n-k}) + [\varphi'(\bar{p}_{n-k+1}) - \varphi'(\bar{p}_{n-k})]S$$

for each $k = 1, \ldots, n-1$. Once we have determined the complete power set $\bar{p}_0 < \bar{p}_1 < \cdots < \bar{p}_n$ we calculate the corresponding switching times from

$$t_{i+1} = \frac{E_0}{\bar{p}_0}\left[1 + \frac{S}{\bar{p}_1} + \cdots + \frac{S^i}{\bar{p}_1\bar{p}_2\cdots\bar{p}_i}\right]$$

for each $i = 0, 1, \ldots, n$. Of course this solution may not be feasible because it is likely that we will not satisfy the constraint $t_{n+1} = T$. However, this problem is overcome by adjusting the value of $\bar{p}_n$ and repeating the above calculations. In this regard note that

$$\omega'(\bar{p}_{n-1})\frac{d\bar{p}_{n-1}}{d\bar{p}_n} = \omega'(\bar{p}_n) - S\varphi''(\bar{p}_n),$$

and hence deduce that

$$\frac{d\bar{p}_{n-1}}{d\bar{p}_n} = \frac{\omega'(\bar{p}_n)}{\omega'(\bar{p}_{n-1})}\left[1 + \frac{S}{\bar{p}_n}\right] > 0.$$

In similar fashion we have

$$\omega'(\bar{p}_{n-2})\frac{d\bar{p}_{n-2}}{d\bar{p}_n} = [\omega'(\bar{p}_{n-1}) - S\varphi''(\bar{p}_{n-1})]\frac{d\bar{p}_{n-1}}{d\bar{p}_n} + S\varphi''(\bar{p}_n),$$

from which it follows that

$$\omega'(\bar{p}_{n-2})\frac{d\bar{p}_{n-2}}{d\bar{p}_n} = \omega'(\bar{p}_{n-1})\left[1 + \frac{S}{\bar{p}_{n-1}}\right]\frac{\omega'(\bar{p}_n)}{\omega'(\bar{p}_{n-1})}\left[1 + \frac{S}{\bar{p}_n}\right] - \omega'(\bar{p}_n)\frac{S}{\bar{p}_n},$$

and hence

$$\frac{d\bar{p}_{n-2}}{d\bar{p}_n} = \frac{\omega'(\bar{p}_n)}{\omega'(\bar{p}_{n-2})}\left[1 + \frac{S}{\bar{p}_{n-1}} + \frac{S^2}{\bar{p}_{n-1}\bar{p}_n}\right] > 0.$$

By continuing in this way we obtain the general formula

$$\frac{d\bar{p}_{n-k}}{d\bar{p}_n} = \frac{\omega'(\bar{p}_n)}{\omega'(\bar{p}_{n-k})}\left[1 + \frac{S}{\bar{p}_{n-k+1}} + \frac{S^2}{\bar{p}_{n-k+1}\bar{p}_{n-k+2}} + \cdots + \frac{S^{n-k}}{\bar{p}_{n-k+1}\bar{p}_{n-k+2}\cdots\bar{p}_n}\right] > 0 \tag{5}$$

for each $k = 1, 2, \ldots, n$. If $\bar{p}_n$ increases see that $\bar{p}_i$ increases for all $i = 0, 1, \ldots, n-1$ and hence t_{n+1} decreases. If $\bar{p}_n$ decreases then t_{n+1} increases.

Theorem 3 *If the battery capacity is infinite and if $s(t) = S$ for all $t \in [0, T]$ then there is a unique value of $\bar{p}_n$ such that $t_{n+1} = T$. In this case the time taken for stage i of the journey is*

$$\Delta t_i = \frac{E_0 S^i}{\bar{p}_0\bar{p}_1\cdots\bar{p}_i} \tag{6}$$

for each $i = 0, \ldots, n$, and the total distance travelled is

$$x_{\max} = \frac{E_0}{\bar{p}_0}\left[\varphi(\bar{p}_0) + \varphi(\bar{p}_1)\frac{S}{\bar{p}_1} + \cdots + \varphi(\bar{p}_n)\frac{S^n}{\bar{p}_1\bar{p}_2\cdots\bar{p}_n}\right]. \tag{7}$$

6 The case in which solar power is constant and the battery capacity is finite

We assume that $s(t) = S$ for all $t \in [0, T]$. In this section we also assume that the battery has finite capacity C and that $E_0 = C$. For an optimal solution we must satisfy the requirements that $S\Delta t_i \le C$ for each $i = 0, \ldots, n-1$. If we assume the necessary conditions for a maximal journey are satisfied then the capacity constraints can be written as

$$\bar{p}_0 \bar{p}_1 \cdots \bar{p}_i \geq S^{i+1} \tag{8}$$

for each $i = 0, \ldots, n-1$. Let us reconsider the necessary conditions. From equation (5) see that as the power $\bar{p}_n$ on the final interval is increased the power $\bar{p}_i$ on every other interval is also increased. If $\bar{p}_n$ is increased until $\bar{p}_0 = S$ then the optimality condition (4) ensures that the capacity constraints (8) are also satisfied. If

$$t_{n+1} = \frac{C}{\bar{p}_0} \left[1 + \frac{S}{\bar{p}_1} + \cdots + \frac{S^n}{\bar{p}_1 \bar{p}_2 \cdots \bar{p}_n} \right] \geq T \,,$$

then we simply increase $\bar{p}_n$ until $t_{n+1} = T$. If $t_{n+1} < T$ then we increase the number of time intervals. Note that the capacity constraint (8) implies that

$$t_{n+1} \geq \frac{(n+1)C}{S} \,.$$

By choosing n sufficiently large we ensure that $t_{n+1} \geq T$. Now we repeat the earlier adjustment procedure to satisfy the time constraint.

7 Results for non-constant solar power

Calculating optimal switching points is more difficult when solar power is not constant. We used a steepest ascent method to find a numerical solution.

Table 1 shows distance travelled, average speed and energy collected for each interval of a journey with hourly switching times. Solar energy is calculated from typical solar irradiance curves for

Table 1: Hourly switching times

time (HH:MM)	dist (km)	speed (km/h)	energy (Wh)
8:00			
9:00	57.81	57.81	1109.3
10:00	37.39	37.39	1418.4
11:00	43.43	43.43	1647.1
12:00	47.32	47.32	1782.5
13:00	49.43	49.43	1816.9
14:00	49.95	49.95	1748.3
15:00	48.91	48.91	1580.7
16:00	46.23	46.23	1323.5
17:00	41.69	41.69	991.3
	422.17	46.91	

Table 2: Optimal switching times

time (HH:MM)	dist (km)	speed (km/h)	energy (Wh)
8:00			
9:38	72.36	44.38	1972.7
10:54	58.21	45.69	2037.7
12:09	58.71	47.13	2220.3
13:26	62.48	48.56	2322.1
14:43	63.82	49.97	2121.5
15:49	56.83	51.51	1534.0
16:34	39.71	53.55	817.8
16:54	19.88	57.29	314.8
17:00	6.39	68.02	77.3
	438.39	48.71	

Central Australia. The total distance travelled is 422.17 km and the average speed is 46.91 km/h.

Table 2 shows the results if we use the same number of interchanges but the batteries are interchanged at the computed optimal times. The original switching times from Table 1 were improved using a steepest-ascent method. The final switching times were found after 1905 steps. On the final step the improvement in total distance travelled was less than 1 metre. The total distance travelled with this strategy is 438.39 km and the average speed is 48.71 km/h.

8 Conclusions

It is interesting to review the report on the 1996 World Solar Challenge [8] and the recent literature on optimal driving strategies for solar powered cars. The pertinent observation in the context of this paper is that a constant speed strategy, where feasible, is essentially the most efficient in terms of energy consumption [3]. However, in general we find that other factors must be considered and in many cases an optimal strategy may require small variations in speed [1, 2, 4, 5, 6, 7].

The optimal strategy proposed in this paper resulted in an increase of 16 km in distance travelled and is clearly significant in the context of a solar car race. We should nevertheless admit that the strategy was not used by Annesley College during the 2001 World Solar Challenge. On the other hand we are pleased to report that two year 12 students from the College did use the above algorithm to calculate improved switching times and to understand the fundamental ideas of optimal control.

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