Practical insight through perturbation analysis

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Abstract

Though industrial processes are perceived to be dauntingly complex from a mathematical modelling perspective, simple mathematical models often provide remarkable insight. One of the reasons behind this apparent contradiction is that the mathematical models often involve small and/or large non-dimensional parameters and therefore are amenable to simplification through the use of perturbation analysis. In many cases, the leading order perturbation approximation provides the bulk of the information about the structure and behaviour of the solution and is often sufficient to obtain profound insight into the phenomenon under examination. This article illustrates such utility of the perturbation analysis by using examples in which leading order perturbation approximations provide substantial insight into the mode transition phenomena in the vibrational behaviour of curved beams and helices. The methodology described in this article can be used in a wide range of applications to reveal the simplest possible structure of the mathematical model that answers the question under examination.

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1 Introduction

> It is the easiest thing in the world to formulate complex mathematical models of real-world phenomenon. What is required, for a good model, is the simplest possible structure that answers the question under examination. *Bob Anderssen, 2004 [1]*

Though industrial processes are perceived to be dauntingly complex from a mathematical modelling perspective, in reality simple mathematical models often provide remarkable insight. The reasons behind this contradiction were examined by de Hoog [3] who suggested that one of the reasons is a robustness of industrial processes, which usually implies that the process is weakly coupled to the environment and therefore only a few effects dominate. As a result, robust processes often can be described by simple mathematical models. The argument in favour of robustness of industrial processes is that they are often similar to the processes that take place in nature. Examples include segregation of particles on the basis of size and specific gravity that takes place on a scree slope and also in a box of cereal. These processes are similar to the processes that take place in gravity separators such as jigs. Another example is crystal growth, which produces spectacular features such as stalagmites, stalactites and shawls in limestone caves, but at the same time is also a key process in the production of alumina, table salt and refined sugar. Further support for robustness is that many industrial processes are quite old and were successfully implemented long before sophisticated computer control was even envisioned. It would simply not have been feasible in the past to implement processes that were not robust.

Another important reason behind the possibility of model simplification is that the mathematical model of industrial processes often involve small and/or large parameters. Typical examples of small and large parameters are aspect ratios, energy ratios and stiffness ratios, which appear in a large variety of industrial processes such as metal rolling, vibration of shells, extrusion, lubrication, and filtration. Problems that contains small or large parameters are particularly amenable to simplification through the use of perturbation analysis.

The focus of this article is on the use of perturbation analysis as a tool for model simplification and for providing an insight into the phenomenon under examination. A brief introduction to perturbation analysis is given in Section 2. A particular advantage of perturbation analysis is that it reveals the simplest possible structure of the mathematical model that still contains all the important features of the problem under examination. Equations associated with this simplified description (which are called leading order perturbation equations) often can be solved in closed form. The resulting analytic expressions not only highlight the essential structure of the solution, but also provide substantial insight into the phenomenon. Thus, in practical applications, a leading order perturbation approximation is often all that is required as it is capable of providing an insightful answer to the question under examination. Such utility of perturbation analysis is examined in this article using three examples from the analysis of the vibration of curved beams (see Section 3). These examples demonstrate how a leading order perturbation approximation has provided insight into the mode transition phenomenon in the vibration of curved beams and has led to the discovery of a new mode transition phenomenon in the vibration of helices. These examples also demonstrate how to examine different regions of the vibrational spectrum and curvature parameter by introducing different small parameters into the same mathematical description of the problem. The role of both regular and singular perturbations is illustrated. Finally, discussion and conclusions are given in Section 4.

2 Perturbation analysis as a tool for model simplification

Perturbation methods are among the most useful and powerful techniques for simplification of models that contain a small parameter (note that a large parameter is easily converted into a small parameter).

Perturbations analysis involves the following three steps [2]:

1. convert the original problem into a perturbation problem by introducing or identifying the small parameter;
2. assume an expression for the solution in the form of a perturbation series and compute the coefficients of that series;
3. recover the solution to the original problem by summing the perturbation series.

The small parameter is not unique for a given problem and needs to be selected to address a question under examination. It is preferable to introduce the small parameter in such a way that the leading order term in the perturbation series can be obtained in a closed form. The examples in Section 3 illustrate how different regions of model parameters are studied through the introduction of different small parameters into the same mathematical model.

The emphasis of this article is on the leading order term in the perturbation series, as this term often provides the bulk of the information about the structure and behaviour of the solution. A leading order perturbation approximation is often sufficient to obtain insight into the phenomenon under examination, while higher order terms improve the accuracy of the approximate solution. Examples in Section 3 illustrate how leading order perturbation approximation provides insight into the phenomenon under investigation and may even lead to the discovery of new phenomena.

There are two types of perturbations problems: *regular perturbation* and *singular perturbation*. *Regular perturbation* problems have a unique solution to the *reduced problem* (the problem obtained by setting the small parameter, say ϵ, to zero), which provides a good approximation to the solution of the original problem when ϵ is small. In contrast, the reduced problem for *singular perturbation* problem normally does not have a unique solution, and the treatment of singular perturbation problems is somewhat more complex. Both singular and regular perturbation problems are illustrated in the examples in the next section. Both types of perturbations can arise from the same original problem through the use of different scaling.

Perturbation solutions are often difficult to justify rigorously. The practical emphasis is on answering the question under examination, and experimental verification is usually required to validate the perturbation results. This is illustrated in the following section.

3 Mode transition phenomena in curved beam vibration

The study described in this section was motivated by a project on the design of a new gas meter. Specifically, the project required an explanation for the high frequency mode transition phenomenon in the vibration of arbitrarily curved beams with clamped ends, that was observed experimentally. The application of perturbation analysis in the high frequency region of the spectrum has provided substantial insight into the mode transition phenomenon, and has motivated the extension of the analysis to other regions of the vibrational spectrum and the curvature parameter. This led to discovery of a new mode transition phenomenon, which occurs with increase in beam opening angle, as the beam gradually transforms into a helix. This discovery has, in turn, motivated an introduction of a new structure, a hyperhelix, with intriguing connection to string theory of elementary particles and with practical applications in biologically inspired robot actuators.

The examples in this section illustrate that leading order perturbation approximations are capable of providing substantial insight into the phenomenon. In these examples, different perturbation models have been formulated by introducing different small parameters into the original mathematical model, which lead to insight into the mode transition phenomena in different regions of the vibrational spectrum and curvature parameter. Both numerical and experimental verifications have been used extensively in the examples in Sections 3.1–3.3, to verify the validity of perturbation approximations.

3.1 High mode number mode transition

The non-dimensional equations of free vibration of an arbitrarily curved beam is an eigenvalue problem for the ordinary differential equations of the form [4] (here, we consider a simplified form of the equations for constant cross-section of a beam)

$$\begin{aligned} -\epsilon\{-[\kappa^2(u'-\kappa v)]' + \kappa[\kappa(u'-\kappa v)]' + [\kappa(v'+\kappa u)']' \\ - \kappa(v'+\kappa u)''\} + (u'-\kappa v)' + \lambda u = 0\,, \end{aligned} \tag{1}$$

$$\begin{aligned} -\epsilon\{-\kappa^3(u'-\kappa v) - [\kappa(u'-\kappa v)]'' + \kappa^2(v'+\kappa u)' \\ + (v'+\kappa u)'''\} + \kappa(u'-\kappa v) + \lambda v = 0\,, \end{aligned} \tag{2}$$

where u and v are the non-dimensional amplitudes of tangential and normal displacements respectively, $\kappa(=\hat{\kappa}l)$ is the non-dimensional curvature function,

l is the length of a beam and $\hat{\kappa}(\bar{s})$ is the curvature of the centre line, λ is the non-dimensional eigenvalue (which is proportional to a squared frequency), $\bar{s}$ is the non-dimensional arc length along the beam centre line $\bar{s} = s/l$. Here and henceforth, prime denotes a differentiation with respect to $\bar{s}$. A small parameter ϵ is

$$\epsilon = h^2/12l^2, \tag{3}$$

where h is the beam thickness. The boundary conditions corresponding to the "clamped" ends are

$$u = v = v' = 0 \quad \text{for } \bar{s} = 0, 1\,. \tag{4}$$

Equations (1), (2) and (4) constitute a *singular perturbation* as the reduced problem (obtained by setting $\epsilon = 0$) consists of a pair of first order differential equations and six boundary conditions. That problem does not have a non-trivial solution. More generally, differential equation problems where the small parameter ϵ multiplies the highest derivatives are singular perturbation problems.

To get an insight into the structure of the solution, it is useful to study the vibration of a beam with constant curvature [4]. Such studies show that the leading order approximation (denoted by a subscript 0) for the eigenfunctions consists of two parts: a slowly varying term and a rapidly varying term, that is,

$$u_0 = u^{(0)}(\bar{s}) + \epsilon^{1/4} u^{(1)}(\bar{s}/\epsilon^{1/4}), \quad v_0 = v^{(0)}(\bar{s}) + v^{(1)}(\bar{s}/\epsilon^{1/4}). \tag{5}$$

The eigenvalues are found by substituting the eigenfunctions into the boundary conditions (4) and requesting that the resulting system has a non-trivial solution (which requires that the determinant of the matrix of the resulting system $\mathbf{D}(\lambda_0)$ is zero). The form of the leading order approximation for eigenfunctions (5) together with the form of the boundary conditions (4) leads to the following result [4]

$$0 = \det \mathbf{D}(\lambda_0) \to \det \mathbf{M}(\lambda_0) \det \mathbf{F}(\lambda_0) \quad \text{as } \epsilon \to 0\,. \tag{6}$$

Thus, as ϵ approaches zero, the leading approximation for the eigenvalue *splits* into the *two asymptotic limits* given by equations

$$\det \mathbf{M}(\lambda_0) = 0\,, \tag{7}$$

and

$$\det \mathbf{F}(\lambda_0) = 0\,. \tag{8}$$

Tarnopolskaya et al. [4] showed that equation (7) describes the eigenvalues of free vibration of a so-called *membrane* (a beam with zero thickness and a given curvature), while equation (8) describes the eigenvalues of *flexural vibrations of a straight beam* (a beam with zero curvature and a given thickness).

Assuming that the form of the eigenfunctions (5) is valid for a general beam curvature function and that the integrals over the beam length of the rapidly varying terms in the eigenfunctions vanish in the limit $\epsilon \to 0$, that is,

$$\int_0^1 u^{(1)}(\bar{s})\,d\bar{s} \to 0\,, \quad \int_0^1 v^{(1)}(\bar{s})\,d\bar{s} \to 0 \quad \text{as } \epsilon \to 0\,, \tag{9}$$

(which is valid for high mode number vibrations), one arrives to the following equations describing the leading order approximations [4]:

$$\phi'' + (\lambda - \bar{\kappa}^2)\phi = 0\,, \tag{10}$$

where $\phi = -(1/\lambda)[(u^{(0)})' - \bar{\kappa} v^{(0)}]$, and

$$\epsilon^{1/4}(u^{(1)})' - \bar{\kappa} v^{(1)} = 0\,, \quad -\epsilon(v^{(1)})''' + \lambda v^{(1)} = 0\,, \tag{11}$$

subject to the boundary conditions

$$\phi' = 0 \quad \text{at } \bar{s} = 0, 1, \tag{12}$$

$$v^{(1)} = -\kappa\phi\,, \quad (v^{(1)})' = \kappa'\phi \quad \text{at } \bar{s} = 0, 1\,. \tag{13}$$

Equations (10) and (12) are a *reduced sub-problem*, which is obtained from (1), (2) and (4) by setting $\epsilon = 0$. It describes the vibrations of a *membrane*.

Equations (11) describe the *flexural vibrations of a straight beam*. The problems of vibration of a membrane and of flexural vibration of a straight beam are *coupled* via the boundary conditions (12, 13).

The existence of the two asymptotic limits for eigenvalues, together with the analysis of the change in eigenfunctions as the non-dimensional curvature parameter increases [4], produces a complete description of a *mode transition phenomenon* in high frequency vibration of curbed beams. The essence of this phenomenon is that, with increase in non-dimensional curvature parameter, some modes undergo the transition from the flexural mode of a straight beam, though an extensional stage, into an inextensional mode, the frequency and mode shape of which is closely approximated by those of the next higher mode of the same symmetry of a straight beam [4, Figures 3 and 4]. During the stage of extensional transition, the frequency is approximated by the frequency of the associated membrane vibration problem, while the mode shape is a superposition of the mode shape of a membrane and the mode shape of flexural vibrations of a straight beam. The exception is the case when the boundary conditions (12, 13) are completely uncoupled. This occurs when $\bar{\kappa} = \bar{\kappa}' = 0$ for $\bar{s} = 0, 1$ and the oscillatory term (mode shape of a flexural vibrations) then disappears as the eigenvalues approach those of a membrane. An example of this was presented by Tarnopolskaya et al. [4].

3.2 Low frequency mode transition

The previous section dealt with the mode transition phenomenon in the high mode number region of spectrum. The analysis has been extended for the low frequency region of the spectrum [6]. In order to do this, a new scaling is required. We consider equations (1, 2) again, and introduce the following scaling [6]

$$\Lambda = \lambda/\epsilon \,, \quad \bar{\kappa} = \kappa/\sqrt{\epsilon} \,, \quad \bar{u} = u/\sqrt{\epsilon} \,, \tag{14}$$

We also introduce a one parameter family of non-dimensional curvature functions of the form

$$\kappa = bK(\bar{s}), \tag{15}$$

where b is the non-dimensional curvature parameter ($b = \hat{b}l$), $\hat{b}$ is the dimensional curvature parameter, and $K(\bar{s})$ is a given curvature function. A new small non-dimensional parameter is

$$\bar{\epsilon} = h\hat{b}/\sqrt{12} \tag{16}$$

(this parameter is small for thin beams with small curvature parameter $\hat{b}$). With the new scaling, equations (1) and (2) become [6]

$$\begin{aligned} &\bar{\epsilon}^2[K^2(\bar{u}' - \bar{\kappa}\nu)]' - \bar{\epsilon}^2K[K(\bar{u}' - \bar{\kappa}\nu)]' - \bar{\epsilon}[K(\nu' + \bar{\epsilon}K\bar{u})']' \\ &\qquad + K\bar{\epsilon}(\nu' + \bar{\epsilon}Ku)'' + (\bar{u}' - \bar{\kappa}\nu)' + \bar{\epsilon}\Lambda\bar{u}/\bar{b} = 0\,, \end{aligned} \tag{17}$$

$$\begin{aligned} &\bar{\epsilon}^2\bar{b}K^3(\bar{u}' - \bar{\kappa}\nu) + \bar{\epsilon}[K(\bar{u}' - \bar{\kappa}\nu)]'' - \bar{\epsilon}\bar{b}K^2(\nu' + \bar{\epsilon}K\bar{u})' \\ &\qquad - (\nu' + \bar{\epsilon}K\bar{u})''' + \bar{\kappa}(\bar{u}' - \bar{\kappa}\nu) + \Lambda\bar{\nu} = 0\,, \end{aligned} \tag{18}$$

where $\bar{b} = b/\sqrt{\epsilon}$. This is now a *regular perturbation* problem. A standard perturbation expansion applies

$$\Lambda(\bar{\epsilon}) = \sum_{k=0}^{\infty} \bar{\epsilon}^k\Lambda_k\,, \quad \bar{u}(\bar{s}, \bar{\epsilon}) = \sum_{k=0}^{\infty} \bar{\epsilon}^k\bar{u}_k(\bar{s}), \quad \nu(\bar{s}, \bar{\epsilon}) = \sum_{k=0}^{\infty} \bar{\epsilon}^k\nu_k(\bar{s}). \tag{19}$$

Tarnopolskaya et al. [6] showed that ν_0 is the solution of

$$-\nu_0'''' + \Lambda_0\nu_0 = \bar{\kappa}\int_0^1 \bar{\kappa}\nu_0\,d\bar{s}\,. \tag{20}$$

If

$$\int_0^1 \bar{\kappa}\nu_0\,d\bar{s} = 0\,, \tag{21}$$

equation (20) describes the *flexural vibration of a straight beam.* Equation (20) provides insight into the mode transition phenomenon. It shows that no mode transition occurs if (21) holds. In this case, the frequency and mode shape of the curved beam are given, up to the zeroth order, by the corresponding functions for a straight beam. For example, in the case when the curvature is an antisymmetric function, the modes symmetric in ν remain unchanged, while the modes antisymmetric in ν undergo mode transition.

The solution of (20) has a form

$$\nu_0 = \nu_{0h} + \nu_{0p} \,, \tag{22}$$

where ν_{0h} is the solution of the problem of free vibration of a straight beam (homogeneous problem)

$$-\nu_{0h}'''' + \Lambda_0 \nu_{0h} = 0 \,, \tag{23}$$

while ν_{0p} is a particular solution of equation (20). For a beam curvature represented by a polynomial of degree up to three, a particular solution is

$$\nu_{0p} = \frac{\bar{\kappa}}{\Lambda_0} \int_0^1 \bar{\kappa} \nu_0 \, d\bar{s} \,. \tag{24}$$

Thus, during the mode transition the mode shape represents the superposition of the mode shape of a straight beam and a component proportional to the beam curvature function. This highlights a main difference between the high mode number and the low frequency mode transition phenomenon: in the former the slowly varying component during the mode transition represents the mode shape of a membrane.

3.3 From beam to helix

The previous two sections discuss the mode transition phenomenon for small values of beam curvature. The same equations can be used to study the vibration of a beam with large curvature. Assuming that the beam curvature is constant, the opening angle α of the beam is introduced instead of curvature function. Once the opening angle becomes greater than 2π, it is impossible to realise a strictly planar uniformly curved beam of this type. However, to an adequate approximation, a beam with large constant curvature is treated as a helix, provided that a helical pitch is small compared with its diameter, and that motion parallel to the helical axis is ruled out.

The equations of free vibrations (1, 2) are, after some algebraic manipulations [5],

$$v'''''' + 2\alpha^2 v'''' + (\alpha^4 - \Lambda)v'' + \Lambda v \alpha^2 + \epsilon[\Lambda(\alpha^4 v + 2\alpha^2 v'' + v'''' - \Lambda v)] = 0\,. \quad (25)$$

Boundary conditions at the clamped ends are [5]

$$v''''' + 2\alpha^2 v''' = 0 \quad \text{for } \bar{s} = 0, 1\,. \quad (26)$$

Differential equation (25) is a *regular perturbation* problem. The leading order approximation is obtained by setting $\epsilon = 0$, that is,

$$v'''''' + 2\alpha^2 v'''' + (\alpha^4 - \Lambda)v'' + \Lambda v \alpha^2 = 0\,. \quad (27)$$

The reduced problem (27) is the familiar equation of *flexural vibration* of a portion of a ring. Numerical results show that this approximation provides sufficient accuracy for a wide range of parameter ϵ. Equation (27) is studied further by considering large values of α and introducing a new small parameter

$$\hat{\epsilon} = \sqrt{\Lambda}/\alpha^2. \quad (28)$$

A general solution for the eigenfunctions has the form

$$v = c_1 f[\beta_1 \alpha(\bar{s} - 1/2)] + c_2 f[\beta_2 \alpha(\bar{s} - 1/2)] + c_3 f[\beta_3 \alpha(\bar{s} - 1/2)], \quad (29)$$

where $f \equiv \cos$ for symmetric modes, and $f \equiv \sin$ for anti-symmetric modes, c_i and β_i, $i = 1, 2, 3$, are the unknown coefficients. Following a standard procedure for determining the eigenvalues (that is, substituting expressions for the eigenfunctions into the boundary conditions and requiring that the resulting system of equations in c_i has a non-trivial solution) leads to an eigenvalue equation which splits into two equations as $\hat{\epsilon} \to 0$ [5]. This leads to the existence of the *two families of the leading order approximations* for the eigenvalues (a phenomenon similar to that observed in Section 3.1).

For symmetric modes, the two families are [5]

$$\text{Family I} \quad \Lambda_k^{I} = 4\alpha^2(\pi k)^2, \quad k = 1, 2, \ldots, \tag{30}$$
$$\text{Family II} \quad \Lambda_k^{II} = 2k^2\pi^2\alpha[\alpha - (-1)^k 2\sin\alpha] + O(1), \quad k = 1, 2, \ldots, \tag{31}$$

while for anti-symmetric modes they are

$$\text{Family I} \quad \Lambda_k^{I} = (\pi\alpha)^2(2k-1)^2, \quad k = 1, 2, \ldots, \tag{32}$$
$$\text{Family II} \quad \Lambda_k^{II} = 2k^2\pi^2\alpha[\alpha + (-1)^k 2\sin\alpha] + O(1), \quad k = 1, 2, \ldots. \tag{33}$$

The eigenvalues as functions of α do not intersect. It is therefore possible to arrange them in ascending order of frequency. This order defines the order in which the modes of a curved beam take one or another type of vibrational behaviour at large opening angle.

While the eigenvalues of the two families are of the same order of magnitude, the mode shapes of the two families are drastically different. For the *first family*, the shape of the tangential displacements, up to the leading order, is of order $O(\alpha)$ and is

$$u = \alpha \sin[2k\pi(\bar{s} - 1/2)], \quad k = 1, 2, \ldots. \tag{34}$$

The leading order part of tangential displacements is a slowly varying function. Tarnopolskaya et al. [5] showed that the transverse displacements are of smaller order than the tangential ones ($O(1)$) . The eigenfunctions of the *second family* have distinctively different shape. Both the tangential and the transverse displacements are highly oscillatory functions with spatial frequency of oscillation proportional to the opening angle, modulated by a slowly varying function with frequency proportional to the mode index k.

The interpretation of the two families was given by Tarnopolskaya et al. [5]. The first family represents the torsional vibration of a helix with respect to helical axis, while the second family represents transverse vibration of a helix with respect to helical axis. It is rather intriguing that the flexural modes of the curved beam transforms, with increase in the opening angle, into such a different types of vibration of a helix!

These results have been extensively tested via a number of experiments designed to verify both the frequencies and the mode shape with increase in the opening angle [5]. All theoretical findings have been confirmed.

An intriguing feature of helix vibration is that, at a macroscopic level, the helix behaves as a straight rod with appropriately low density and elastic modulus, while the vibrations of a curved beam are hidden.

This work inspired the introduction of a *hyperhelix* by Fletcher et al. [7]. A hyperhelix of order N is a helix coiled up into a helix, coiled up into a helix, and so on until this process is repeated N times (in a self-similar regression that makes the hyperhelix a fractal object). The hyperhelix has the essential ability to hide dimensions and mode details within an apparently simple structure, which suggests an analogy with string theory of elementary particles, in which hidden dimensions are coiled up invisibly within simple structures resembling strings or spirals [8].

The properties of hyperhelices appear to be useful in practice due to strain amplification, that occurs because of the hierarchical structure, and ability to produce specific output displacements [9, 10]. They find application in biologically inspired robot actuators [9, 10].

4 Conclusions

We illustrated the importance of perturbation analysis in highlighting the essential structure of the solution and providing a profound insight into the phenomenon under examination.

Three examples demonstrated that substantial insight into the phenomena gained through the leading order perturbation approximation. Numerical and experimental verifications are needed, not only because of the difficulties of the rigorous justification of the validity of the perturbation approximations, but also because a number of simplifications that are usually involved into the formulation of the model.

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