A high order algorithm for ordinary boundary value problems

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Abstract

The method of analytic continuation has been used to obtain numerical solutions of nonlinear initial value problems. Here we formulate the problem in terms of characteristic functions that form a Partition of Unity. This method allows us to extend an approximate local solution to a global solution that can be expressed in closed form as a polynomial whose coefficients are piecewise constant functions. An error bound is calculated and used to prove uniform convergence to the exact solution as the number of partition points approaches infinity. Using a shooting method we also find approximate closed form solutions of nonlinear boundary value problems with arbitrarily high order of accuracy. The solution obtained is useful when a closed form expression is required.

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Contents

1 Introduction

We wish to find approximate solutions in closed form of the nonlinear system of differential equations of order M

$$\frac{dy}{dx} = f(x, y)\,, \tag{1}$$

for x on an interval $I = [a, b]$ with initial or boundary conditions and with arbitrarily high order of accuracy. The function $f(x, y)$ is a vector function $f : I \times \mathbb{R}^M \mapsto \mathbb{R}^M$ that is assumed to be analytic in a domain D of $I \times \mathbb{R}^M$ and y is an M-dimensional vector.

One technique used to develop a high order algorithm for the initial value problem (IVP) involves the use of an iterative method based on analytic

continuation [1, 2]. This method has been used to obtain numerical solutions of (1) subject to the initial condition $y(x_0) = y_0$ [3].

In this article we reformulate the problem in terms of characteristic functions that constitute a Partition of Unity. This allows us to extend a local solution to the whole interval $[a, b]$ and also perform an error and convergence analysis of the problem with a considerable simplification of the algebra involved.

Viera [4] used characteristic functions to find approximate closed form solutions of linear boundary value problems in an infinite domain. In that case, the method also yields a closed form expression for the eigenvalue equation. In this article, characteristic functions are used to obtain the solution of nonlinear IVPs in closed form and then use a shooting method to extend the technique to solve nonlinear boundary value problems (BVP).

As an illustration of how the technique is applied, we use the Taylor series method to calculate the local solutions. The approximate global solution is then written in closed form as a single polynomial whose coefficients are piecewise constant functions of x.

2 Initial value problem

2.1 Problem formulation

A partition of $I = [a, b]$ is introduced with $N + 1$ points and N subintervals $I_n = [x_n, x_{n+1})$ given by

$$p_N : a = x_0 < x_1 < \cdots < x_N = b\,.$$

The partition norm is defined by $|p_N| = \max_{n=0,1,\ldots,N-1} |x_{n+1} - x_n|$ so that $|p_N| \to 0$ as $N \to \infty$. Introduce the characteristic function of the interval I_n

defined by

$$\chi_n = \begin{cases} 1, & \text{if } x \in I_n\,, \\ 0, & \text{otherwise.} \end{cases}$$

The set of functions χ_n form a Partition of Unity where $\sum_{n=0}^{N-1} \chi_n = 1$ for each $x \in I$. They also form an orthogonal set with respect to real multiplication,

$$\chi_n \chi_m = \begin{cases} 0, & n \neq m\,, \\ \chi_n, & n = m\,. \end{cases} \tag{2}$$

These properties are used to extend a local solution globally to the interval $[a, b]$ as follows.

Let $y = \psi(x; x_0)$ be a local solution of (1) valid in a neighbourhood of $x = x_0$. This is called the initial local solution. Introduce the piecewise constant function

$$x^N = \sum_{n=0}^{N-1} x_n \chi_n\,. \tag{3}$$

The Partition of Unity then allows us to extend the initial solution globally by evaluating it at $x_0 = x^N$, thus,

$$\begin{aligned} y &= \psi(x; x^N) \\ &= \psi\left(x; \textstyle\sum_{n=0}^{N-1} x_n \chi_n\right) \\ &= \sum_{n=0}^{N-1} \psi(x; x_n)\chi_n\,. \end{aligned} \tag{4}$$

The equality of the last two expressions in (4) is easily established by taking a value of $x \in I_j$. Then the two expressions become equal to $\psi(x; x_j)$ by the properties of χ_j mentioned above. To ensure continuity of the solution at the partition points we impose the condition

$$\psi(x_n; x_{n-1}) = \psi(x_n; x_n)\,, \quad n = 1, 2, \ldots, N-1\,. \tag{5}$$

Equation (4) has two main implications: the global solution on $[a, b]$ is represented

1. as the piecewise superposition of components, each of which is valid on the interval I_n, for $n = 0, 1, \ldots, N-1$; or
2. as a single function, whose parameter x_0 has been replaced by a piecewise constant function x^N that approaches the continuous variable x as $N \to \infty$.

If the initial local solution $\psi(x; x_0)$ is expressed in terms of a power series and if the radius of convergence of the series defined on I_n is $R_n \geq |x_{n+1} - x_n|$ for $n = 0, 1, \ldots, N-1$, then (4) and (5) are equivalent to applying the method of analytic continuation to extend the initial solution globally to $[a, b]$. The formulation using characteristic functions, however, provides an effective tool that allows application of the method to more difficult problems such as finding solutions of nonlinear partial differential equations.

2.2 Local solution

In order to illustrate how the characteristic functions are used to express a global solution, we calculate the local solutions using the Taylor series method. Let the Taylor expansion of the initial solution $\psi(x; x_0)$ about $x = x_0$ be

$$\psi(x; x_0) = \psi_K(x; x_0) + T_K(x; x_0, \xi_0) \,, \tag{6}$$

where

$$\psi_K(x; x_0) = \sum_{k=0}^{K} \frac{\psi^{(k)}(x_0)}{k!} (x - x_0)^k \,, \tag{7}$$

$$T_K(x; x_0, \xi_0) = \frac{\psi^{(K+1)}(\xi_0)}{(K+1)!} (x - x_0)^{K+1}, \tag{8}$$

are the Taylor polynomial of degree K and the corresponding Lagrange form of the remainder, respectively, and $\xi_0 \in (x_0, x) \subset I_0$. The derivatives in the expansion are determined by successive differentiation of equation (1):

$$\begin{aligned}
\psi^{(0)}(x_0) &= y_0\,, \\
\psi^{(1)}(x_0) &= f(x_0, y_0) \\
\psi^{(2)}(x_0) &= \Big[\frac{\partial f}{\partial x} + f(x,y)\frac{\partial f}{\partial y}\Big]_{(x_0,y_0)} \\
&\vdots \\
\psi^{(K+1)}(x_0) &= \Big[\Big(\frac{\partial}{\partial x} + f(x,y)\frac{\partial}{\partial y}\Big)^K f(x,y)\Big]_{(x_0,y_0)}\,.
\end{aligned} \tag{9}$$

Cauchy used the method of Majorants to calculate bounds for the coefficients and establish the following conditions under which the series solution converges.

If the function $f(x, y)$ is analytic in the neighbourhood of the point (x_0, y_0), equation (1) has a unique solution $y = y(x)$ which is analytic in the neighbourhood of x_0 and satisfies the initial condition $y(x_0) = y_0$, [1, 2].

2.3 Global solution

Neglecting the remainder term we approximate the local solution (6) by $\psi(x; x_0) \approx \psi_K(x; x_0)$. Replacing $x_0 = x^N$, the approximate global solution ψ_{NK} is written in the form

$$\begin{aligned}
\psi_{NK}(x) &\equiv \psi_K(x; x^N) \\
&= \psi_K\left(x; \textstyle\sum_{n=0}^{N-1} x_n \chi_n\right) \\
&= \displaystyle\sum_{n=0}^{N-1} \psi_K(x; x_n)\chi_n\,,
\end{aligned} \tag{10}$$

where

$$\psi_K(x;x_n) = \sum_{k=0}^{K} \frac{a_{nk}}{k!}(x-x_n)^k\,, \quad a_{nk} = \psi^{(k)}(x_n)\,. \tag{11}$$

The continuity condition (5) becomes

$$\psi_K(x_n;x_{n-1}) = \psi_K(x_n;x_n)\,, \quad n = 1,2,\ldots,N-1\,.$$

If required, the expression for $\psi_{NK}(x)$ in (10) is written in a more compact form as a single polynomial of degree K

$$\begin{aligned}\psi_{NK}(x) \equiv& \psi_K(x;x^N)\\ =& \sum_{k=0}^{K} \frac{\psi^{(k)}(x^N)}{k!}(x-x^N)^k\\ =& \alpha_0^N + \alpha_1^N x + \cdots + \alpha_K^N x^K\,,\end{aligned} \tag{12}$$

where α_j^N for $j = 0,1,\ldots,K$, are piecewise constant functions of x. Note that $x^N \to x$ as $N \to \infty$. Hence, $\alpha_j^N \to 0$ as $N \to \infty$ for $j = 1,2,\ldots,K$, and α_0^N approaches the exact solution in that limit.

3 Error bound and convergence

In order to calculate a uniform bound for the error, we globalize the remainder term (8) by applying to it the same process we used to obtain (10). By analogy with (3) we introduce the sequence of intermediate values

$$\xi^N = \sum_{i=0}^{N-1} \xi_i \chi_i\,,$$

where $\xi_i \in (x_i,x) \subset I_i$ and evaluate (8) at $(\xi_0;x_0) = (\xi^N;x^N)$. The vector absolute error is then defined by

$$E_{NK}(x) = T_K(x;x^N,\xi^N)$$

$$
\begin{aligned}
&= \frac{\psi^{(K+1)}(\xi^N)}{(K+1)!}(x - x^N)^{K+1} \\
&= \frac{\psi^{(K+1)}\left(\sum_i \xi_i \chi_i\right)}{(K+1)!}(x - \textstyle\sum_n x_n \chi_n)^{K+1} \\
&= \sum_i \sum_n \frac{\psi^{(K+1)}(\xi_i)}{(K+1)!}(x - x_n)^{K+1} \chi_i \chi_n \\
&= \sum_{n=0}^{N-1} \frac{\psi^{(K+1)}(\xi_n)}{(K+1)!}(x - x_n)^{K+1} \chi_n \,,
\end{aligned}
$$

where the orthogonality (2) of χ_n is used to reduce the double summation to a single sum in n. Taking the Euclidean norm gives the scalar error

$$
\begin{aligned}
e_{NK}(x) &= \|E^{NK}(x)\| \\
&\leq \frac{1}{(K+1)!} \sum_{n=0}^{N-1} \left\|\psi^{(K+1)}(\xi_n)\right\| (x - x_n)^{K+1} \chi_n \\
&\leq \frac{|p_N|^K}{(K+1)!} S_{NK} \,,
\end{aligned} \tag{13}
$$

where $|p_N|$ is the partition norm and

$$
S_{NK} = \sum_{n=0}^{N-1} \left\|\psi^{(K+1)}(\xi_n)\right\| (x_{n+1} - x_n) \geq 0
$$

is a Riemann sum. Since by the Cauchy theorem mentioned in Section 2.2 the solution is an analytic function, the derivative $\left\|\psi^{(K+1)}(x)\right\|$ is continuous and bounded. Defining

$$
M_K = \max_{x \in [a,b]} \left\|\psi^{(K+1)}(x)\right\| ,
$$

gives

$$
S_{NK} \leq M_K \sum_{n=0}^{N-1} (x_{n+1} - x_n) = M_K (x_N - x_0) \,,
$$

and the following error bound is obtained

$$e_{NK}(x) \le B_{NK}, \quad B_{NK} = \frac{|p_N|^K}{(K+1)!} M_K (x_N - x_0). \tag{14}$$

Hence, $e_{NK}(x) \to 0$ uniformly and the approximate solution converges uniformly to the exact solution with convergence rate $O(|p_N|^K)$ as $N \to \infty$ for fixed K.

4 Boundary value problem

4.1 Shooting method

We now use the shooting method to solve the second order nonlinear boundary value problem

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad x \in [a, b], \quad y(a) = \alpha\,, \quad y(b) = \beta\,, \tag{15}$$

by replacing it with an equivalent IVP [6] of the form

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad x \in [a, b], \quad y(a) = \alpha\,, \quad y'(a) = t\,. \tag{16}$$

Recall that the second order equation can always be written as a system of equations of the form (1). If $y(x, t)$ is the solution of (16) and if $t = t_0$, where t_0 is the solution of the nonlinear transcendental equation

$$y(b, t_0) - \beta = 0\,, \tag{17}$$

then the IVP (16) is equivalent to the BVP (15). Some conditions ensure existence and uniqueness of the solution of the BVP (15) [5, e.g.].

TABLE 1: L_2 norm of absolute error $e_{NK}(x)$ and error bound B_{NK} for $K = 10$ and several values of the partition size N. For comparison with Table 2, the last column shows approximate values of execution time in seconds.

N	$\|e_{NK}(x)\|_{L_2}$	B_{NK}	cpu
10	0.40×10^{-13}	0.16×10^{-11}	15
50	0.15×10^{-19}	0.17×10^{-18}	50
100	0.27×10^{-22}	0.16×10^{-21}	90
200	0.52×10^{-25}	0.16×10^{-24}	210
400	0.10×10^{-27}	0.72×10^{-27}	450

4.2 Example

We now find the approximate solution ψ_{NK} of the boundary value problem

$$\frac{d^2y}{dx^2} = y^3 - yy', \quad x \in [1,2], \quad y(1) = 1/2, \quad y(2) = 1/3, \tag{18}$$

and compare it with the exact solution $\psi(x) = 1/(x+1)$. The solution of (17) with $b = 2$ and $\beta = 1/3$ is found using the method of false position giving a value of $t_0 = -0.24999\ldots$. It is not difficult to automate the search of two initial guesses of opposite sign to be used by false position to compute the root to the high level of accuracy required by the method.

Table 1 shows the L_2 norm of the actual error $e_{NK}(x) = |\psi(x) - \psi_{NK}(x)|$, using the exact solution $\psi(x)$. The table also shows the error bound B_{NK} calculated using (14) for $K = 10$ and several values of the partition size N. Table 2 gives the L_2 norm of e_{NK} and the bound B_{NK} for $N = 10$ and several values of the Taylor polynomial order K.

For comparison, the last column in both tables shows approximate execution time in seconds. When $(N, K) = (200, 10)$ (Table 1), the error bound $B_{NK} = O(10^{-24})$. This computation takes about 210 seconds to execute.

TABLE 2: L_2 norm of absolute error $e_{NK}(x)$ and error bound B_{NK} for $N = 10$ ($\Delta x = 0.1$) and several values of the Taylor polynomial order K. For comparison with Table 1, the last column shows approximate values of execution time in seconds.

K	$\|e_{NK}(x)\|_{L_2}$	B_{NK}	cpu
5	0.11×10^{-6}	0.17×10^{-5}	10
10	0.40×10^{-13}	0.16×10^{-11}	15
20	0.39×10^{-26}	0.55×10^{-24}	30
40	0.45×10^{-52}	0.19×10^{-49}	100
80	0.38×10^{-102}	0.69×10^{-101}	430

When $(N, K) = (10, 20)$ (Table 2), the error bound has the same order of magnitude but the computation takes only about 30 seconds to complete (seven times faster).

On the other hand, the computations in the last row of each table, both take approximately the same amount of time to execute (≈ 450 sec). However, in Table 1 the error is $O(10^{-27})$ whereas in Table 2 the error is $O(10^{-101})$, a substantial improvement. This behaviour is due to the exponential dependence of B_{NK} on the parameter K in equation (14) and the $(K + 1)!$ term in the denominator.

5 Discussion

We have formulated the calculation of approximate solutions of nonlinear initial and boundary value problems in terms of the characteristic functions χ_n of the interval $[a, b]$. These functions form a Partition of Unity of $[a, b]$ and hence are used to extend a local solution valid in the neighbourhood of $x = a$ to the whole interval $[a, b]$.

The technique not only allows us to obtain (approximate) solutions in closed form but error analysis and convergence calculations are considerably simplified, as shown by the analysis of Section 4. This technique has not been used previously in the context of ordinary nonlinear differential equations.

As an example, we calculated the local solutions using the Taylor series method. We then used a specific differential equation to compute explicitly the approximate solution and compare with the known exact solution. We emphasize that other ways of computing local solutions could be used just as well. Work is currently in progress to extend the method to solve initial value problems for partial differential equations.

The following pseudocode uses the Taylor series method to calculate the solution of the second order IVP

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad x \in [a, b], \quad y(a) = y_0, \quad y'(a) = y'_0\,,$$

and is implemented using a computer algebra system.

Input: a, b, $f(x, y, y')$, y_0, y'_0, N, K
Output: Approximate solution

$$\psi_{NK}(x) = \sum_{n=0}^{N-1} \sum_{k=0}^{K} \frac{a_{nk}}{k!} (x - x_n)^k \chi_n$$

Set $\Delta x = (b - a)/N$
Calculate derivatives

$$\psi^{(k)}[x, y(x)] = \left(\frac{\partial}{\partial x} + f(x, y) \frac{\partial}{\partial y} \right)^{k-1} [f(x, y)], \quad k = 1, \ldots, (K + 1)$$

Initialize first two coefficients: $a_{00} = y_0\,,\ a_{01} = y'_0$
for $n = 0, \ldots, (N - 1)$ **do**
  for $k = 2, \ldots, (K + 1)$ **do**
    Calculate derivatives at $(x, y) = (x_n, y_n)\,,\ a_{nk} = \psi^{(k)}[x_n, y_n]$

7: **end for**
8: Calculate local solutions

$$\psi_K(x;x_n) = \sum_{k=0}^{K} \frac{a_{nk}}{k!}(x-x_n)^k$$

$$\frac{d\psi_K}{dx}(x;x_n) = \sum_{k=1}^{K} \frac{a_{nk}}{(k-1)!}(x-x_n)^{k-1}$$

9: Update partition points and function values $x_{n+1} = x_n + \Delta x$, $y_{n+1} = \psi_K(x_{n+1};x_n)$, $y'_{n+1} = \frac{d\psi_K}{dx}(x_{n+1};x_n)$.
10: Update first two coefficients to ensure continuity of global solution: $a_{(n+1)0} = y_{n+1}$, $a_{(n+1)1} = y'_{n+1}$
11: **end for**
12: Construct final solution:

$$\psi_{NK}(x) = \sum_{n=0}^{N-1} \psi_K(x;x_n)\chi_n$$

References

[1] E. T. Copson, *Introduction to the Theory of Functions of a Complex Variable*, Oxford University Press, 1970. C993, C996

[2] G. Birkhoff and G. Rota, *Ordinary Differential Equations*, Wiley, 1978. C993, C996

[3] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, 1962. C993

[4] F. Viera, On approximate closed form solutions of linear ordinary differential equations, *ANZIAM J.* **47** (EMAC2005) pp.C245–C260, 2006. http://anziamj.austms.org.au/V47EMAC2005/Viera C993

[5] H. B. Keller, *Numerical Methods for Two-Point Boundary Value Problems*, Blaisdell, Waltham, Mass, 1968. C999

[6] R. L. Burden and J. D. Faires, *Numerical Analysis*, PWS Publishing Company, Boston, 1993. C999

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