A mathematical model for heat transfer in grain store microclimates

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Abstract

Australia's reputation as a supplier of insect-free grain is being threatened by Psocids (*Liposcelis* spp.), an insect pest which is wreaking havoc within the Australian grain industry. These pests are very mobile and appear to move in and out of infested grain bulks in response to variations in temperature. This movement is the cause of much difficulty in controlling these insects so an understanding of what

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happens to the heat transfer at the surface of the grain bulk would allow a better understanding of the observed behaviour by these insects. Here we examine the heat transfer at the grain store surface and the grain bulk surface. A heat transfer variant of the theory of "double-diffusivity" is developed, which is a mathematical model that assumes two separate diffusion paths; one for high-diffusivity and one for regular-diffusivity. This approach takes into consideration the fact that the rate of heat transfer through the grain is different to that through the interstitial air surrounding the grain. Based on a heat-balance approach, approximate analytical results are obtained from which the overall variation in temperature close to the grain store wall may be calculated. The behaviour for typical parameter values is shown graphically.

Contents

1 Introduction

Australia is one of the worlds major grain exporters with a reputation for supplying clean, high-quality, and most important of all, insect-free grain. In recent years Psocids have proved to be a very problematic pest to the Australian grain industry. One reason why these particular pests are a major problem is because of their ability to move rapidly in and out of infested grain bulks (up to approximately 30 mh^{-1}) in, what is believed to be, a response to changes in environmental factors, and in particular temperature. This movement makes it quite difficult to control the populations of these insects by pest control methods such as fumigation as their movement means that a number of them do not remain long enough within the fumigated grain bulk to absorb a lethal dosage of the fumigant. An understanding of what happens from a heat transfer viewpoint at the surface of the grain bulk will allow a better understanding of the observed behaviour by these insects. It will allow predictions to be made of insect activity which may lead to application of pest control methods to be timed to periods of certain insect activity. The surface of the grain bulk which comes into contact with the vertical grain store wall is of particular interest as this is where considerable diurnal temperature fluctuations occur and where Psocids have been observed.

In this study we examine the heat transfer on the microscopic scale at the grain bulk surface which comes into contact with the vertical grain store surface and we ignore any convective effects. Our aim is to understand the micro-environment which the insect experiences in such a region, and to determine how ambient temperatures affect the temperature at this surface. We examine this "micro-surface" by developing a mathematical model to obtain a temperature profile of this region. This model is based on a heat transfer variant of the theory of "double-diffusivity" as proposed by Hill [1]. Such an approach lends itself well to our problem as it takes into consideration the fact that the rate of heat transfer through the grain is different to that through the interstitial air surrounding the grain.

In the following section we propose modelling the flow of heat through a grain silo by means of the established mathematical model of diffusion in a media with two distinct diffusivity mechanisms, one of high-diffusivity and one of regular-diffusivity, and referred to as "double-diffusivity" theory. Here we propose a heat transfer variant of the "double-diffusivity" model in which heat is propagated along both grain-paths and air-paths, and in addition there is a transfer of heat from grains to air and vice-versa. In the subsequent section we describe a heat-balance approximate solution of the coupled partial differential equations (2). The coupled ordinary differential equations (21) and (22) for the two moving fronts $X_1(t)$ and $X_2(t)$ are solved numerically and some results are presented in Section 4.

2 The double-diffusivity model

Following the notion of double-diffusivity as outlined in Hill [1] and related papers [2]–[6], we envisage a closely packed grain silo such that every point in the silo is connected to every other point by either a grain-path or by an air-path, and we envisage heat propagating along both air and grain paths. Moreover, we make the idealisation that at every point we can associate an air-temperature $T_1(x,t)$ and a grain-temperature $T_2(x,t)$ such that the actual physical temperature $T(x,t)$ is determined from

$$T(x,t) = \frac{1}{2}\left(T_1(x,t) + T_2(x,t)\right). \tag{1}$$

In addition, for heat propagating along air and grain paths, we speculate that there can be transfer of heat from grains to air, and vice-versa. As described in detail in [1], the one-dimensional double-diffusivity model takes the form

$$\begin{aligned} \frac{\partial T_1}{\partial t} &= \kappa_1 \frac{\partial^2 T_1}{\partial x^2} - k_1 T_1(x,t) + k_2 T_2(x,t), \\ \frac{\partial T_2}{\partial t} &= \kappa_2 \frac{\partial^2 T_2}{\partial x^2} + k_1 T_1(x,t) - k_2 T_2(x,t), \end{aligned} \tag{2}$$

where κ_1 and κ_2 denote the thermal diffusivities of air and grain respectively, and k_1 and k_2 denote the heat-transfer coefficients from air to grain, and from grain to air respectively. This version of the double-diffusivity model was first proposed by Rubinstein [7], and from the random walk derivation given in [1] we have $k_1, k_2 > 0$. We wish to solve (2) subject to the following initial and boundary conditions

$$T_1(x,0) = T_{10}, \quad T_2(x,0) = T_{20}, \tag{3}$$

$$T_1(0,t) = T_b(t), \quad T_2(0,t) = T_b(t), \tag{4}$$

where T_{10} and T_{20} are constant interior solutions of (2) and therefore satisfy

$$k_1 T_{10} = k_2 T_{20}, \tag{5}$$

and $T_b(t)$ is some prescribed time-dependent boundary temperature. In the following section we propose a heat-balance approximate solution of (2)–(4) using time-dependent cubic profiles in the spatial variable x for both $T_1(x,t)$ and $T_2(x,t)$.

3 The heat-balance integral method

The basic idea of the heat-balance integral method for the ordinary heat conduction equation is to assume a "penetration depth", which varies with time and beyond which there is effectively no heat-flow. Such an approach is well suited to our problem as such penetration depths are observed in grain silos. In addition a simple polynomial expression in the spatial variable is adopted with all the necessary boundary data and the assumed conditions applying at the time varying penetration front. An ordinary differential equation for the moving penetration front is then determined by satisfying the partial differential equation in an average or integral sense. In the case of the coupled system (2) we assume two moving penetration fronts $X_1(t)$ and $X_2(t)$. For this problem we deduce a coupled system of ordinary differential equations for $X_1(t)$ and $X_2(t)$ as follows.

For $T_1(x,t)$ we suppose that for $x \geq X_1(t)$, we have

$$T_1(x,t) = T_{10}, \quad \frac{\partial T_1}{\partial x}(x,t) = 0, \tag{6}$$

so that in particular on the moving heat front $x = X_1(t)$, we have

$$T_1(X_1(t),t) = T_{10}, \quad \frac{\partial T_1}{\partial x}(X_1(t),t) = 0. \tag{7}$$

Upon differentiating the first of these equations with respect to time and utilising the second equation we may deduce

$$\frac{\partial T_1}{\partial t}(X_1(t),t) = 0, \tag{8}$$

and accordingly from $(2)_1$, we have

$$\frac{\partial^2 T_1}{\partial x^2}(X_1(t),t) = \frac{k_1}{\kappa_1}T_{10} - \frac{k_2}{\kappa_1}T_2(X_1(t),t). \tag{9}$$

It is a simple matter to show that the general time-dependent cubic temperature profile

$$T_1(x,t) = a(t) + b(t)x + c(t)x^2 + d(t)x^3, \tag{10}$$

satisfying (4)$_1$, (7) and (9) becomes

$$\begin{aligned} T_1(x,t) &= T_{10} + (T_b(t) - T_{10})\left(1 - \frac{x}{X_1(t)}\right)^3 \\ &\quad - (T_2(X_1(t),t) - T_{20})\,\frac{X_1(t)xk_2}{2\kappa_1}\left(1 - \frac{x}{X_1(t)}\right)^2, \end{aligned} \tag{11}$$

upon noting the relation (5).

Similarly, for $T_2(x,t)$ we have

$$T_2(X_2(t),t) = T_{20}, \quad \frac{\partial T_2}{\partial x}(X_2(t),t) = 0, \quad \frac{\partial^2 T_2}{\partial x^2}(X_2(t),t) = \frac{k_2}{\kappa_2}T_{20} - \frac{k_1}{\kappa_2}T_1(X_2(t),t), \tag{12}$$

and the most general cubic profile satisfying these conditions and (4)$_2$ and (5) can be shown to become

$$\begin{aligned} T_2(x,t) &= T_{20} + (T_b(t) - T_{20})\left(1 - \frac{x}{X_2(t)}\right)^3 \\ &\quad - (T_1(X_2(t),t) - T_{10})\,\frac{X_2(t)xk_1}{2\kappa_2}\left(1 - \frac{x}{X_2(t)}\right)^2. \end{aligned} \tag{13}$$

Now, on introducing the averages $\theta_1(t)$ and $\theta_2(t)$ defined by

$$\theta_1(t) = \int_0^{X_1(t)} T_1(x,t)dx, \quad \theta_2(t) = \int_0^{X_2(t)} T_2(x,t)dx, \tag{14}$$

and integrating $(2)_1$ over $[0, X_1(t)]$ and $(2)_2$ over $[0, X_2(t)]$ we may deduce the following equations

$$
\begin{aligned}
\frac{d\theta_1(t)}{dt} + k_1\theta_1 &= -\kappa_1\frac{\partial T_1}{\partial x}(0,t) + \frac{dX_1}{dt}T_{10} + k_2\int_0^{X_1(t)} T_2(x,t)dx, \\
\frac{d\theta_2(t)}{dt} + k_2\theta_2 &= -\kappa_2\frac{\partial T_2}{\partial x}(0,t) + \frac{dX_2}{dt}T_{20} + k_1\int_0^{X_2(t)} T_1(x,t)dx.
\end{aligned}
\tag{15}
$$

In order to evaluate the integrals on the right-hand side of $(15)_1$ we use the approximate profiles (11) and (13) to obtain

$$
\begin{aligned}
\int_0^{X_1(t)} T_2(x,t)\,dx &= T_b(t)X_1(t) + X_1(t)\left(T_b(t) - T_{20}\right) \\
&\quad \times \left(\frac{X_1(t)^2}{X_2(t)^2} - \frac{1}{4}\frac{X_1(t)^3}{X_2(t)^3} - \frac{3}{2}\frac{X_1(t)}{X_2(t)}\right) \\
&\quad - \frac{k_1X_2(t)X_1(t)^2}{2\kappa_2}\left(T_1(X_2(t),t) - T_{10}\right) \\
&\quad \times \left(\frac{1}{2} - \frac{2}{3}\frac{X_1(t)}{X_2(t)} + \frac{1}{4}\frac{X_1(t)^2}{X_2(t)^2}\right),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\int_0^{X_2(t)} T_1(x,t)\,dx &= T_b(t)X_2(t) + X_2(t)\left(T_b(t) - T_{10}\right) \\
&\quad \times \left(\frac{X_2(t)^2}{X_1(t)^2} - \frac{1}{4}\frac{X_2(t)^3}{X_1(t)^3} - \frac{3}{2}\frac{X_2(t)}{X_1(t)}\right)
\end{aligned}
$$

$$-\frac{k_2 X_1(t) X_2(t)^2}{2\kappa_1}\left(T_2(X_1(t),t)-T_{20}\right)$$
$$\times\left(\frac{1}{2}-\frac{2}{3}\frac{X_2(t)}{X_1(t)}+\frac{1}{4}\frac{X_2(t)^2}{X_1(t)^2}\right). \tag{17}$$

Further, on using the following expressions obtained from the cubic temperature profiles (11) and (13) we find

$$\begin{aligned}
\theta_1(t) &= T_{10}X_1(t)+\frac{X_1(t)}{4}\left(T_b(t)-T_{10}\right)-\frac{X_1(t)^3 k_2}{24\kappa_1}\left(T_2(X_1(t),t)-T_{20}\right),\\
\frac{\partial T_1}{\partial x}(0,t) &= \frac{-3}{X_1(t)}\left(T_b(t)-T_{10}\right)-\frac{X_1(t)k_2}{2\kappa_1}\left(T_2(X_1(t),t)-T_{20}\right),\\
\theta_2(t) &= T_{20}X_2(t)+\frac{X_2(t)}{4}\left(T_b(t)-T_{20}\right)-\frac{X_2(t)^3 k_1}{24\kappa_2}\left(T_1(X_2(t),t)-T_{10}\right),\\
\frac{\partial T_2}{\partial x}(0,t) &= \frac{-3}{X_2(t)}\left(T_b(t)-T_{20}\right)-\frac{X_2(t)k_1}{2\kappa_2}\left(T_1(X_2(t),t)-T_{10}\right),
\end{aligned} \tag{18}$$

and from (5), (15) and (16)–(18) we may deduce the coupled ordinary differential equations for $X_1(t)$ and $X_2(t)$:

$$X_1(t)\frac{\partial}{\partial t}\left(T_b(t)\right)+X_1'(t)\left(T_b(t)-T_{10}\right)-\frac{X_1(t)^2X_1'(t)k_2}{2\kappa_1}\left(T_2(X_1(t),t)-T_{20}\right)$$
$$-\frac{X_1(t)^3k_2}{6\kappa_1}\frac{\partial}{\partial t}\left(T_2(X_1(t),t)\right)=\left(T_b(t)-T_{10}\right)\left(\frac{12\kappa_1}{X_1(t)}-X_1(t)k_1\right)$$

$$
\begin{aligned}
&+ \left(T_2(X_1(t),t) - T_{20}\right)\left(1 + \frac{X_1(t)^2 k_1}{12\kappa_1}\right) 2X_1(t)k_2 \\
&+ \left(T_b(t) - T_{20}\right) 4X_1(t)k_2 \left(1 - \frac{3}{2}\frac{X_1(t)}{X_2(t)} + \frac{X_1(t)^2}{X_2(t)^2} - \frac{1}{4}\frac{X_1(t)^3}{X_2(t)^3}\right) \\
&+ \left(T_1(X_2(t),t) - T_{10}\right) \frac{4k_1k_2X_1(t)^2}{\kappa_2}\left(\frac{X_1(t)}{3} - \frac{X_2(t)}{4} - \frac{X_1(t)^2}{8X_2(t)}\right), \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
&X_2(t)\frac{\partial}{\partial t}\left(T_b(t)\right) + X_2'(t)\left(T_b(t) - T_{20}\right) - \frac{X_2(t)^2 X_2'(t) k_1}{2\kappa_2}\left(T_1(X_2(t),t) - T_{10}\right) \\
&-\frac{X_2(t)^3 k_1}{6\kappa_2}\frac{\partial}{\partial t}\left(T_1(X_2(t),t)\right) = \left(T_b(t) - T_{20}\right)\left(\frac{12\kappa_2}{X_2(t)} - X_2(t)k_2\right) \\
&+ \left(T_1(X_2(t),t) - T_{10}\right)\left(1 + \frac{X_2(t)^2 k_2}{12\kappa_2}\right) 2X_2(t)k_1 \\
&+ \left(T_b(t) - T_{10}\right) 4X_2(t)k_1 \left(1 - \frac{3}{2}\frac{X_2(t)}{X_1(t)} + \frac{X_2(t)^2}{X_1(t)^2} - \frac{1}{4}\frac{X_2(t)^3}{X_1(t)^3}\right) \\
&+ \left(T_2(X_1(t),t) - T_{20}\right) \frac{4k_2k_1X_2(t)^2}{\kappa_1}\left(\frac{X_2(t)}{3} - \frac{X_1(t)}{4} - \frac{X_2(t)^2}{8X_1(t)}\right). \qquad (20)
\end{aligned}
$$

Now, as the thermal diffusivity of air is greater than that of grain, we make the assumption that $X_1(t) > X_2(t)$. This assumption implies $T_2(X_1(t),t) = T_{20}$ which simplifies equations (19) and (20) to

$$
X_1(t)\frac{\partial}{\partial t}\left(T_b(t)\right) + X_1'(t)\left(T_b(t) - T_{10}\right) = \left(T_b(t) - T_{10}\right)\left(\frac{12\kappa_1}{X_1(t)} - X_1(t)k_1\right)
$$

$$+\left(T_b(t)-T_{20}\right)4X_1(t)k_2\left(1-\frac{3}{2}\frac{X_1(t)}{X_2(t)}+\frac{X_1(t)^2}{X_2(t)^2}-\frac{1}{4}\frac{X_1(t)^3}{X_2(t)^3}\right)$$
$$+\left(T_1(X_2(t),t)-T_{10}\right)\frac{4k_1k_2X_1(t)^2}{\kappa_2}\left(\frac{X_1(t)}{3}-\frac{X_2(t)}{4}-\frac{X_1(t)^2}{8X_2(t)}\right), \qquad (21)$$

$$X_2(t)\frac{\partial}{\partial t}\left(T_b(t)\right)+X_2'(t)\left(T_b(t)-T_{20}\right)-\frac{X_2(t)^2X_2'(t)k_1}{2\kappa_2}\left(T_1(X_2(t),t)-T_{10}\right)$$
$$-\frac{X_2(t)^3k_1}{6\kappa_2}\frac{\partial}{\partial t}\left(T_1(X_2(t),t)\right)=\left(T_b(t)-T_{20}\right)\left(\frac{12\kappa_2}{X_2(t)}-X_2(t)k_2\right)$$
$$+\left(T_1(X_2(t),t)-T_{10}\right)\left(1+\frac{X_2(t)^2k_2}{12\kappa_2}\right)2X_2(t)k_1$$
$$+\left(T_b(t)-T_{10}\right)4X_2(t)k_1\left(1-\frac{3}{2}\frac{X_2(t)}{X_1(t)}+\frac{X_2(t)^2}{X_1(t)^2}-\frac{1}{4}\frac{X_2(t)^3}{X_1(t)^3}\right), \qquad (22)$$

which we solve subject to the initial conditions

$$X_1(0)=0, \quad X_2(0)=0. \qquad (23)$$

The numerical solution of (21), (22) and (23) is presented in the following section.

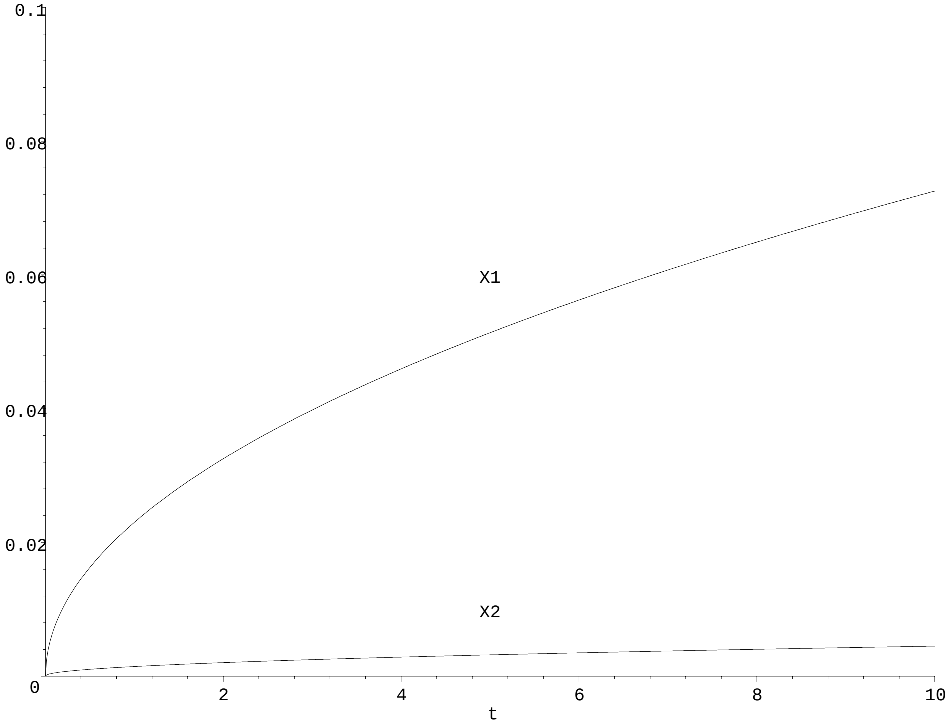

Figure 1: Penetration depths $X_1(t)$ and $X_2(t)$ as a function of time t

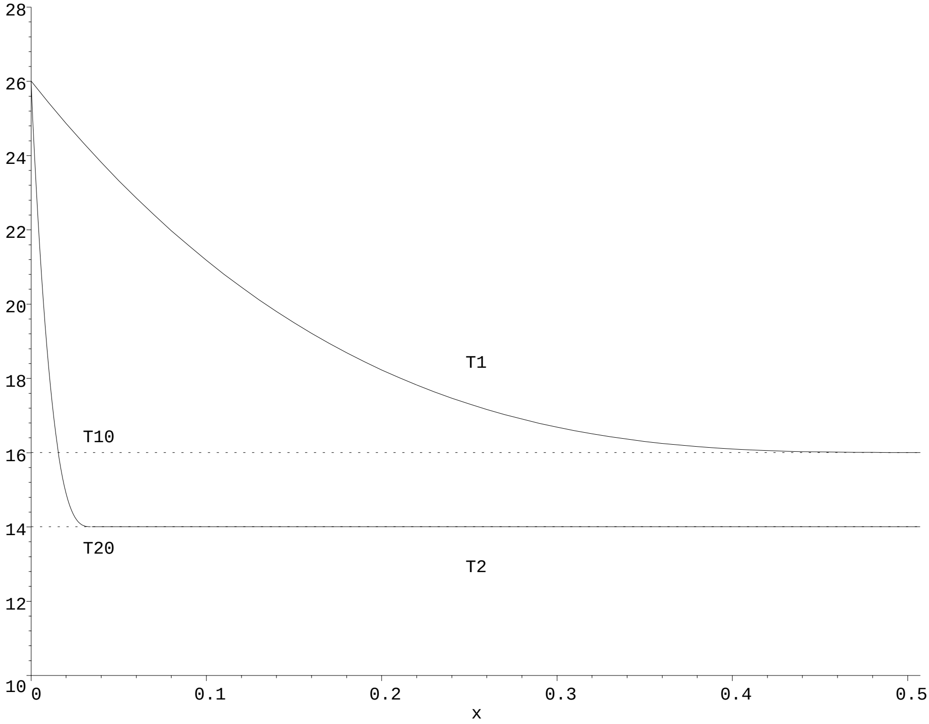

FIGURE 2: Temperatures $T_1(x,t)$ and $T_2(x,t)$ as a function of space x

4 Numerical solution

The numerical results shown in Figures 1 and 2 were obtained using the Fehlberg fourth-fifth order Runge-Kutta method from Maple V Release 5. Figure 1 illustrates the variation in penetration depths $X_1(t)$ and $X_2(t)$ with time for the first 10 seconds for the case of constant boundary temperature $T_b(t) = 26°\mathrm{C}$ at $x = 0$ and the following values of the constants:

$$\begin{aligned} &T_{10} = 16°\mathrm{C}, T_{20} = 14°\mathrm{C}, \kappa_1 = 2.2 \times 10^{-5}\mathrm{m}^2\mathrm{s}^{-1}, \kappa_2 = 8.3 \times 10^{-8}\mathrm{m}^2\mathrm{s}^{-1}, \\ &k_1 = 1 \times 10^{-7}\mathrm{ms}^{-1}, k_2 = 1.14 \times 10^{-7}\mathrm{ms}^{-1}. \end{aligned} \tag{24}$$

We note that κ_1 and κ_2 are standard values of the thermal diffusivities for grain and air while the values of k_1 and k_2 are only crude estimates which are deduced by estimating from the formulae given in Hill [1], namely $s_1 = \lambda^* k_1(\delta x)^2$, $s_2 = \lambda^* k_2(\delta x)^2$, where s_1 and s_2 are probabilities such that $0 \leq s_1, s_2 \leq 1$ and λ^* is the assumed limiting value of $(\delta t)/(\delta x)^2$. Ideally in the future these values might be obtained by using the approximate solution obtained here along with experimental data. At present such experimental data is not available. We note that the penetration depth into the air $X_1(t)$ is larger than that through the grain $X_2(t)$. The reason for this is that the specific heat and density of grain is higher than that of air and hence most of the heat passing through the grain is stored whereas most of the heat passing through the air is conducted along. For the same parameters Figure 2 illustrates the variation of temperature with distance at $t = 600s$. We note that the difference between $T_1(x,t)$ and $T_2(x,t)$ is quite noticeable for distances close to the grain store wall ($x = 0$).

5 Conclusions

An understanding of the heat flow in stored grain silos is important from many practical perspectives, but particularly in following the movement of insect pests. Here we have proposed using the double-diffusivity model which takes into account the fact that the rate of heat transfer through the grain is less than that through the interstitial air surrounding the grain. The approximate analytical solution obtained via the Heat-Balance Integral Method gives overall agreement with observations made in the field for penetration depths and temperature profiles, but exact agreement is not known until experimental data can be obtained.

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