Hybrid algorithms for cyclically reduced convection-diffusion problems

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Abstract

We consider hybrid and adaptive iterative algorithms for cyclically-reduced discrete convection-diffusion problems. Hybrid algorithms combine via a two phase algorithm, iterative methods which require no a priori information about the coefficient matrix in the first phase with Chebyshev or Richardson iteration in the second phase. For

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two-dimensional convection-diffusion problems, central difference discretization is considered and the resulting linear system is reduced to approximately half its size by applying one step of cyclic reduction. We examine the numerical performance of the hybrid methods for solving the reduced systems. Our numerical experiments show that for the class of problems considered, an adaptive Chebyshev algorithm that uses modified moments to approximate the eigenvalues requires less work in most cases than the hybrid algorithms based on GMRES/Richardson methods.

Contents

1 Introduction

The GMRES method [13] with restarting is often used for solving nonsymmetric linear systems. An effect of restarting is that it slows down the convergence as information about the smallest eigenvalues and corresponding eigenvectors is lost at the time of restart. Morgan [9] has proposed the addition of approximate eigenvectors to improve the convergence. Another alternative to improve convergence is to use a hybrid algorithm which combines GMRES with Richardson or Chebyshev methods [10].

In this paper, we study the performance of some hybrid iterative algorithms for solving linear systems arising from cyclically reduced discrete convection-diffusion problems. For solving a linear system with coefficient matrix A, a hybrid method consists of two phases: a first phase which acquires eigenvalue information about the matrix A and a second phase which uses the eigenvalue information to construct a polynomial iteration.

Consider the convection-diffusion equation

$$-\left[(pu_x)_x + (qu_y)_y\right] + su_x + tu_y = f \quad \text{on} \quad \Omega = (0,1)\times(0,1) \tag{1}$$

with $u = g$ on $\partial\Omega$. The functions $p(x,y)$ and $q(x,y)$ are positive in Ω. We discretize (1) by centred five-point finite difference scheme over a uniform grid of mesh size $h = 1/(n+1)$. The resulting linear system using a point red-black ordering of the underlying grid has the form [4]

$$\begin{pmatrix} D & C \\ E & F \end{pmatrix} \begin{pmatrix} u_h^{(r)} \\ u_h^{(b)} \end{pmatrix} = \begin{pmatrix} f_h^{(r)} \\ f_h^{(b)} \end{pmatrix}, \tag{2}$$

where D and F are nonsingular diagonal matrices. Eliminating the red points gives the reduced system

$$\left[F - ED^{-1}C\right] u_h^{(b)} = f_h^{(b)} - ED^{-1} f_h^{(r)}. \tag{3}$$

For the iterative solution of (3), Elman and Golub [3] considered one-line and two-line orderings of the grid points. The resulting coefficient matrix $A^{(b)} = F - ED^{-1}C$ derived using each of these orderings have block Property A and using the analysis of Gauss-Seidel and SOR methods for solving the constant coefficient problems $p(x, y) = q(x, y) = 1$, $s(x, y) = \sigma$, and $t(x, y) = \delta$ they showed that the rates of convergence for solving the reduced system by block iterative methods were faster than for solving the full system.

2 Hybrid Procedures

Manteuffel [7, 8] first proposed a hybrid algorithm based on the Chebyshev polynomials in the complex plane for solving nonsymmetric linear systems whose spectrums can be enclosed in an ellipse that does not contain the origin. For the nonsymmetric linear system

$$Au = b, \quad A \in \mathrm{R}^{N \times N}, \quad u, b \in \mathrm{R}^N, \tag{4}$$

Phase I of Manteuffel's algorithm computes the convex hull of the spectrum of A and in Phase II, the iteration parameters d, the centre and c, the focal length of the smallest ellipse with foci at $d \pm c$ enclosing the spectrum are used in a Chebyshev iteration.

Let

$$T_j(z) = \cosh(j \cosh^{-1}(z))$$

be the j^{th} Chebyshev polynomial and let p_j be the scaled Chebyshev polynomial given by

$$p_j(\lambda) = \frac{T_j(\frac{d-\lambda}{c})}{T_j(\frac{d}{c})},$$

with the parameters c and d restricted so that $d > 0$ and $d^2 > c^2$. Denoting by u_0, an initial approximate solution of (4), the iterates u_j in Phase II of Manteuffel's algorithm are such that

$$e_j = p_j(A)e_0 \quad \text{and} \quad r_j = p_j(A)r_0, \quad j \geq 0, \tag{5}$$

where $e_j = A^{-1}b - u_j$ denotes the error in u_j, $r_j = b - Au_j$ is the residual vector. Starting from an initial guess u_0 and iteration parameters c and d, Phase II of Manteuffel's algorithm is as follows [7]:

Algorithm 1: Chebyshev Iteration

(1). Compute $r_0 = b - Au_0$ and $p_0 = (1/d)r_0$.
(2). For $j = 1, 2, \ldots,$ until convergence do:
$u_j = u_{j-1} + p_{j-1}$,
$r_j = b - Au_j$,
If $j = 1$, $\alpha_1 = \frac{2d}{2d^2 - c^2}$, $\beta_1 = d\alpha_1 - 1$,
else $\alpha_j = [d - (\frac{c}{2})^2 \alpha_{j-1}]^{-1}$, $\beta_j = d\alpha_j - 1$.
(3). $p_j = \alpha_j r_j + \beta_j p_{j-1}$.

For any choice of the parameters c and d, the rate of convergence of the Chebyshev iteration is given by [7, 11]

$$-\log\left(\max_{\lambda\in\sigma(A)} S(\lambda)\right), \tag{6}$$

where

$$S(z) = \frac{d - z + \left[(d-z)^2 - c^2\right]^{1/2}}{d + \left[d^2 - c^2\right]^{1/2}}. \tag{7}$$

In practice the Chebyshev iteration is performed for a predetermined number of steps. If convergence in Algorithm I is unsatisfactory, the sequence of residual vectors are used to compute estimates of the extreme eigenvalues of A and Algorithm I is restarted using new iteration parameters c and d. An algorithm for estimating the parameters dynamically is given by Manteuffel [8].

3 Eigenvalue Estimates Using Modified Moments

The operator $S(A)$ induced by $S(z)$, given by (7) is such that

$$r_j \approx S(A)^j r_0.$$

In Manteuffel's algorithm, the eigenvalue estimates of A are computed by applying the power method to $S(A)$. Another approach presented in [5] uses modified moments to approximate the eigenvalues of A. For a diagonalizable matrix A, the idea is to interpret the inner products of residual vectors as modified moments and the use of the modified Chebyshev algorithm to compute the entries of a tridiagonal matrix H_κ whose eigenvalues are estimates of the eigenvalues of the matrix A (see [5] for details).

After each $2\kappa - 1$ iterations by the Chebyshev method, a new matrix H_κ is computed and its spectrum is determined. Let $\mathcal{S}(H)$ denote the union of sets of eigenvalues of all the computed matrices H_κ. Then the parameters c and d are determined by fitting an ellipse to $\mathcal{S}(H)$. If parameters d and c change insignificantly when eigenvalues of new matrices H_k are included in the set $\mathcal{S}(H)$, then no further modified moments are calculated.

4 Arnoldi-Based Hybrid Procedures

An alternative to the eigenvalue computation using the modified power method is to use Arnoldi's algorithm [1]. An advantage of this procedure is that the closely related GMRES algorithm [13] can be used to updatethe solution. The spectral information obtained from Arnoldi's algorithm can be used to adaptively build a preconditioner for GMRES [6]. Burrage and Erhel [2] show that the augmented subspace approach of Morgan [9] combined with the preconditioning approach lead to a more robust algorithm.

The spectral information from Arnoldi's algorithm can also be used to obtain a hybrid algorithm. Starting with an initial guess u_0, the hybrid Arnoldi/Chebyshev algorithm can be described as follows [11]:

Algorithm 3: Hybrid Arnoldi/Chebyshev.

(1). Compute $r_0 = b - Au_0$, $\beta = \|r_0\|_2$ and set $v_1 = r_0/\beta$.
(2). **Adaptive Step**:
Perform m steps of Arnoldi/GMRES,
i.e., compute the improved solution $\widehat{u}$ from u_0 and set $u_0 = \widehat{u}$.
Compute eigenvalue estimates of upper Hessenberg matrix
generated by Arnoldi's method and determine new
iteration parameters c and d.
(3).**Chebyshev steps**:

Perform k steps of Algorithm 1 using u_0 and check for convergence. If convergence criteria is not satisfied, go to (1).

A serious drawback of the Chebyshev iteration is that for non-positive real A, the computed convex hull may have eigenvalues with positive and negative real parts [12]. Furthermore, there are problems for which the spectrum is not well approximated by an ellipse. Smolarski and Saylor [14] suggested using nonelliptical contours and used a discrete least-squares approximation on a polygonal region containing eigenvalue estimates to obtain a residual polynomial. Instead of the Chebyshev iteration, in the second phase the residual polynomial is used by means of a cyclic Richardson iteration. However, the method proposed by Smolarski and Saylor to compute the least squares polynomial is based on classical moments and is unstable. The reason for this instability is that the moment matrix $\{\langle \lambda^{i-1}, \lambda^{j-1} \rangle\}$ can become ill-conditioned. Saad [12] replaced the moment matrix by the matrix $\{\langle T_i, T_j \rangle\}$, where T_j is the jth Chebyshev polynomial. Once the modified moment matrix is computed, Saad applies a second-order Richardson iteration based on a polynomial that is optimal in the L^2-sense on some polygon constructed from the eigenvalue estimates.

4.1 Hybrid GMRES Method

All the hybrid algorithms discussed above construct a domain enclosing the eigenvalues in the complex plane and then calculate a residual polynomial.

However, for a nonnormal matrix, eigenvalues alone fail to explain the behaviour of the matrix. Nachtigal, Reichel and Trefethen [10] have noted that there are problems associated with the use of eigenvalue estimates obtained using Arnoldi's algorithm. The hybrid procedure that they have proposed uses the GMRES iteration of Phase I and the residual polynomial used in Phase II of the algorithm is precisely the GMRES polynomial obtained at some step m. Starting with an initial approximation u_0, the hybrid GMRES algorithm can be described as follows:

Algorithm 4: Hybrid GMRES Algorithm.

(1) Phase I:
Run GMRES until $\|r_m\|_2$ drops by a suitable amount and set $\nu = m$.
(2) Phase II:
Re-apply the GMRES polynomial $p_\nu(z)$ cyclically until convergence.

The strategy described in [10] for switching between Phase I and Phase II is that m is chosen such that there are equal amounts of work in Phase I and Phase II.

5 Experimental Results

In this section we describe numerical experiments to analyse the performance of the different hybrid methods for solving the reduced system (3) arising from the central-difference discretisation of convection-diffusion problems of the type (1) on the unit square. The problems considered are taken from [4]. We discretize each problem on the unit square with mesh size $h = 1/(n+1)$. The full linear system has $N = n^2$ equations and the reduced linear system has $N_r = N/4$ equations for even n. Details of the experiments are as follows. We have used MATLAB programs with stopping criterion $\|r_i\|_2/\|r_0\|_2 \leq 10^{-10}$. For all problems, the initial guess is $x_0 = 0$. For the adaptive Chebyshev methods, the adaptive procedure is invoked after at most 16 Chebyshev steps. For the hybrid GMRES algorithm, we have switched to Phase II of the algorithm after 16 GMRES steps. Each experiment examines five methods:

1. CHEBLS: the hybrid least squares method [12],
2. HYBRID GMRES: the hybrid GMRES algorithm [10],
3. CHEBA: the hybrid Chebyshev-GMRES method [11],
4. MP: Adaptive Chebyshev algorithm using modified power method to obtain eigenvalue estimates [7],
5. MM: Adaptive Chebyshev algorithm using modified moments to obtain eigenvalue estimates [5].

In the tables, the numbers of iterations required for solving the reduced linear system and the ILU(0) preconditioned reduced linear system are denoted It and Itp, respectively. The column labelled Flops give the flop counts necessary for satisfying the stopping criterion. In the figures, the convergence histories of the different methods are denoted as follows: A-GMRES(16), B-CHEBLS, C-HYBRID-GMRES, D-CHEBA, E-MP, and F-MM, respectively.

Problem 1. We consider the constant coefficient problem

$$-\Delta u + \sigma u_x = 0, \quad (x, y) \in (0, 1) \times (0, 1), \tag{8}$$

with Dirichlet boundary conditions determined from the exact solution

$$u(x, y) = e^{\sigma x/2} \frac{\sin \pi y}{\sinh \xi} \left(2e^{-\sigma/2} \sinh \xi x + \sinh \xi (1 - x)\right), \tag{9}$$

where $\xi^2 = \pi^2 + \sigma^2/4$. We consider a grid size with $n = 50$. Table 1 shows the number of iterations and the flop counts required to satisfy the stopping criterion. For the Chebyshev method based on modified moments, we have used $\kappa = 4$.

The performance of the methods for the case $\sigma = 500$ is shown in Fig. 1. We remark that the adaptive algorithm MM requires lesser work than the other methods to achieve convergence. The Chebyshev algorithm based on modified moments uses 5 adaptive steps. In this case, the adaptive Chebyshev algorithm based on the power method (MP) uses 4 adaptive steps. We also remark the restarted GMRES algorithm with $m = 16$ converges slowly.

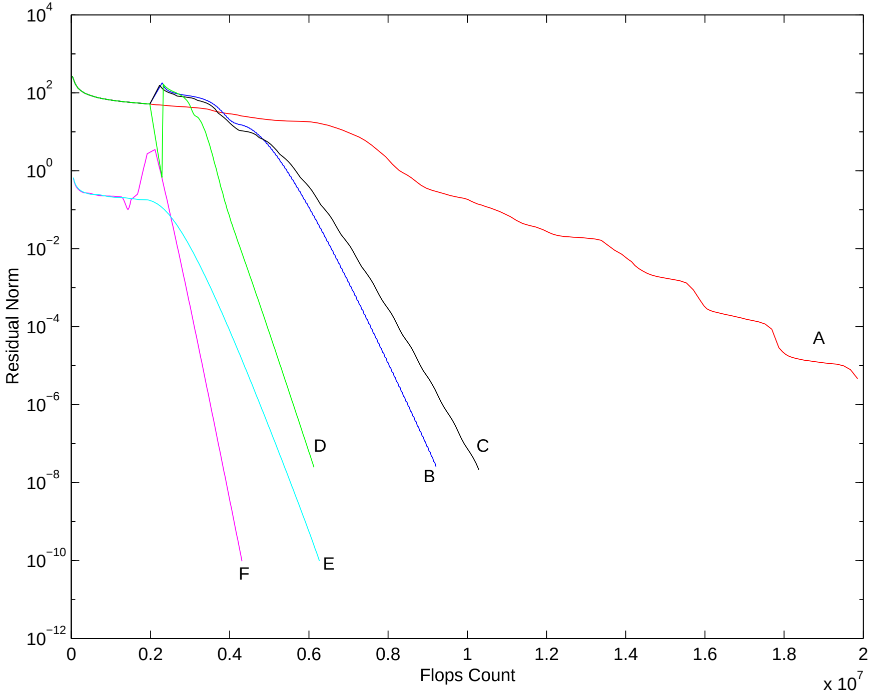

FIGURE 1: Problem 1: $\sigma = 500$

TABLE 1: Iteration and Flop counts ($\times 10^6$) for constant coefficient Problem 1

	$\sigma = 100$		$\sigma = 200$		$\sigma = 500$	
	It	Flops	It	Flops	It	Flops
CHEBLS	107	4.82	163	6.47	254	9.24
HYBRID GMRES	111	9.12	106	8.76	127	10.32
CHEBA	92	4.20	180	6.57	161	6.16
MP	210	6.36	130	4.23	211	6.30
MM	125	3.74	102	3.20	133	4.34

Problem 2. We consider the problem

$$-\Delta u + \sigma u_x + \delta u_y = f(x, y), \quad (x, y) \in (0, 1) \times (0, 1) \tag{10}$$

with Dirichlet boundary conditions $u = 0$. The number of iterations and the flop counts required for convergence are shown in Table 2. We remark that for the highly nonsymmetric case $\sigma = 200$, $\delta = 100$, the hybrid Chebyshev-GMRES method performs well for the preconditioned reduced linear system. For the unpreconditioned linear system, the method CHEBLS performs poorly and its performance is greatly improved in the preconditioned case.

Problem 3. We consider the variable coefficient problem

$$-\Delta u + \frac{\sigma}{2}\left(1 + x^2\right) u_x + \delta u_y = f(x, y), \quad (x, y) \in (0, 1) \times (0, 1) \tag{11}$$

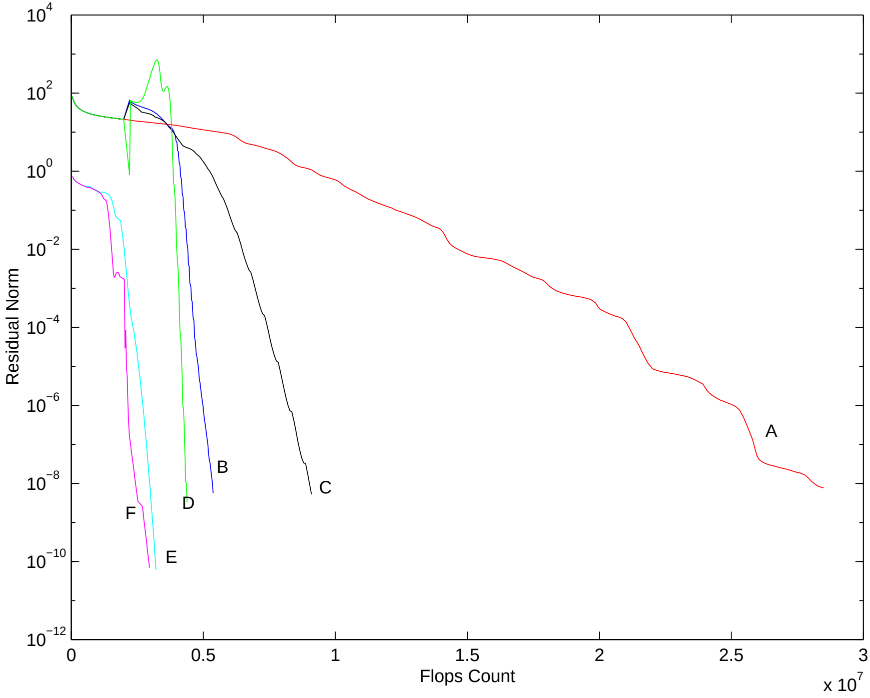

FIGURE 2: Problem 2: $\sigma = 200$, $\delta = 100$

Table 2: Iteration and Flop counts ($\times 10^6$) for Problem 2

	$\sigma = 100,\ \delta = 10$				$\sigma = 200, \delta = 100$			
	It	Flops	Itp	Flops	it	Flops	Itp	Flops
CHEBLS	114	3.96	183	10.53	126	5.40	17	1.17
HYBRID GMRES	85	6.49	84	4.92	111	9.12	47	2.85
CHEBA	83	2.87	98	5.48	99	4.41	12	0.87
MP	80	2.31	39	2.45	99	3.24	32	1.97
MM	62	1.88	38	2.19	91	2.99	22	1.34

with Dirichlet boundary conditions $u = 0$. The discrete solution to the reduced linear system is taken as a vector of ones. We consider a problem size of $n = 50$. Table 3 shows that for the case $\sigma = \delta = 20$, CHEBA requires the least amount of work for the unpreconditioned reduced linear system in this case, whereas the two adaptive methods require a large number of iterations to converge. We also remark that, for $\sigma = \delta = 40$, the hybrid GMRES(16) algorithm performs better than the other methods.

Problem 4. We consider the variable coefficient problem

$$-\Delta u + \sigma(1 - 2x)u_x + \delta(1 - 2y)u_y = 0 \quad (x, y) \in (0, 1) \times (0, 1) \qquad (12)$$

with Dirichlet boundary conditions $u = 0$. Table 4 shows the performance of the different algorithms for solving the reduced linear system.

Table 3: Iteration and Flop counts ($\times 10^6$) for Problem 3

	$\sigma = \delta = 20$				$\sigma = \delta = 40$			
	It	Flops	Itp	Flops	It	Flops	Itp	Flops
CHEBLS	756	23.65	50	3.48	312	9.71	74	4.51
HYBRID GMRES	446	14.23	81	4.85	263	8.28	32	2.05
CHEBA	222	6.86	59	3.87	232	6.78	46	2.81
MP	800	23.07	47	2.74	335	10.04	41	2.37
MM	484	13.93	48	2.79	210	6.18	38	2.24

Table 4: Iteration and Flop counts ($\times 10^6$) for Problem 4

	$\sigma = \delta = 40$				$\sigma = \delta = 60$			
	It	Flops	Itp	Flops	It	Flops	Itp	Flops
CHEBLS	290	6.74	91	3.45	183	4.68	74	4.51
HYBRID GMRES	104	5.58	35	1.41	94	5.10	30	1.23
CHEBA	101	2.95	53	1.99	76	2.51	34	1.34
MP	219	4.26	22	0.90	125	2.58	16	0.62
MM	114	2.16	21	0.81	73	1.48	18	0.69

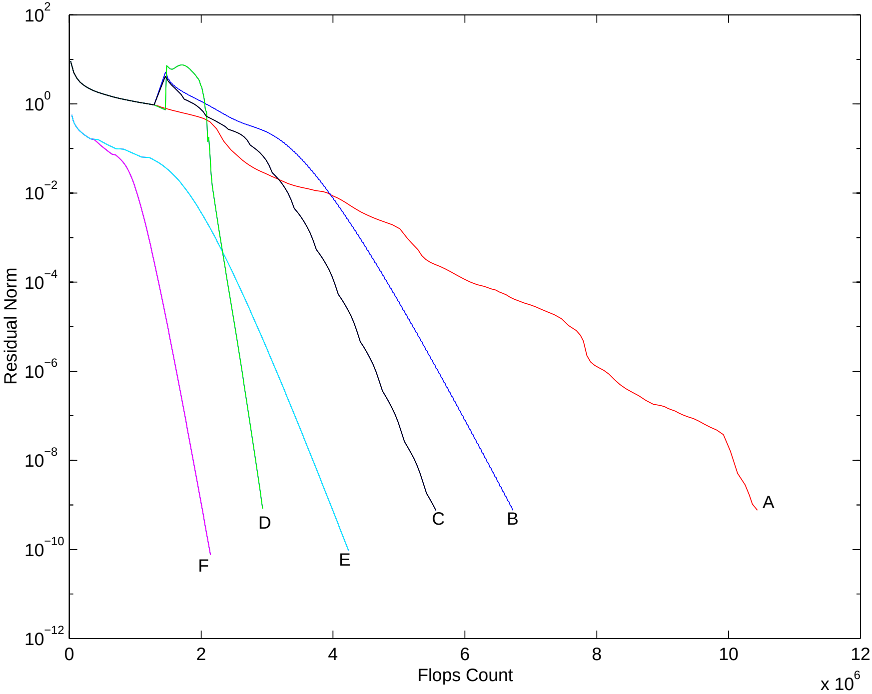

FIGURE 3: Problem 4: $\sigma = \delta = 40$

6 Conclusion

We have carried out an experimental study of the performance of hybrid algorithms for solving reduced linear systems arising from convection-diffusion problems. We have investigated several procedures for obtaining eigenvalue estimates to compute iteration parameters for a polynomial iteration. For the class of problems that we have considered the adaptive Chebyshev scheme based on modified moments often converges faster than the scheme based on the modified power method. Comparisons with other hybrid schemes also show that the Chebyshev algorithm using modified moments requires less work in most cases.

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