

Recovery of the boundary data for a linear second order elliptic problem with a nonlocal boundary condition

Hennie De Schepper* Marián Slodička*

(Received 6 June 2000)

Abstract

We study a linear elliptic partial differential equation of second order in a bounded domain $\Omega \subset \mathbb{R}^N$, with nonstandard boundary conditions on a part Γ of the boundary $\partial\Omega$. Here, neither the solution nor its normal derivative are prescribed pointwise. Instead, the average of the solution over Γ is given and the normal derivative along

*Department of Mathematical Analysis, Ghent University, Galglaan 2, B-9000 Ghent, BELGIUM. <mailto:hds@cage.rug.ac.be> and <mailto:ms@cage.rug.ac.be>

⁰See <http://anziamj.austms.org.au/V42/CTAC99/Desc> for this article and ancillary services, © Austral. Mathematical Soc. 2000. Published 27 Nov 2000.

Γ has to follow a prescribed shape function, apart from an additive (unknown) constant. We prove the well-posedness of the problem and provide a method for the recovery of the unknown boundary data.

Contents

1	Introduction	C519
2	Problem setting	C521
3	Uniqueness and existence	C523
4	Numerical experiments	C528
4.1	First example	C528
4.2	Second example	C531
	References	C535

1 Introduction

The problem of parameter identification from nonstandard boundary conditions (BCs) in boundary value problems (BVPs), originating from various engineering disciplines, is of growing interest. Standard (Dirichlet, Neumann

and Robin type) BCs which are prescribed pointwise are not always adequate, as it depends on the physical context which data can be measured at the boundary of the physical domain. In some cases, it is not possible to prescribe the solution u (pressure, temperature, ...) or the flux $\mathbf{q} \cdot \boldsymbol{\nu}$ pointwise, because only the average value of the solution or of the total flux can be measured along some boundary part Γ . In case that $\int_{\Gamma} u$ or $\int_{\Gamma} \mathbf{q} \cdot \boldsymbol{\nu}$ is prescribed, we are dealing with a nonlocal BC. This type of BC is not sufficient in order to determine the solution uniquely. One needs some additional information, resulting from other physical arguments, e.g., about the shape of u or of $\mathbf{q} \cdot \boldsymbol{\nu}$ along Γ . For instance, nonstandard BCs of the type

$$u = c \in \mathbb{R} \text{ (unknown) on } \Gamma, \quad \int_{\Gamma} -K \nabla u \cdot \boldsymbol{\nu} = s \text{ (given)}, \quad (1)$$

appear in the modelling of soil venting (cf. [2]). Here u stands for the squared air pressure, K denotes the air-transmissivity of the soil matrix, and Γ represents the boundary of an "extraction well" or pump. The BCs can be interpreted as follows. On one hand, the total amount of extracted air can be measured; on the other hand, one can assume the air pressure along the boundary of an extraction well to be constant, however unknown. The existence of a solution for BVPs with BCs including the type (1) has been proved in [2].

The dual type of BCs, viz

$$\mathbf{q} \cdot \boldsymbol{\nu} = c \in \mathbb{R} \text{ (unknown) on } \Gamma, \quad \int_{\Gamma} u = U \text{ (given)}, \quad (2)$$

appears in the modelling of some electromagnetic field problems in electric machinery (cf. [4]). Moreover, it also has a physical interpretation in the above mentioned context of soil venting. For a single well experiment in a homogeneous soil matrix one may assume that the inflow of air into the well tube is constant, but unknown, while the average pressure at the well is given. Here, some pressure variations along the boundary of the well are allowed for.

The main goal of this paper is to study a class of 2nd order linear elliptic PDEs with BCs of a type including (2). We prove the existence and the uniqueness of the solution, and we outline a method for constructing it in terms of the solution of some auxiliary BVPs with *standard* (i.e. pointwise) BCs. In particular, the unknown flux and the trace of the solution at Γ are determined.

2 Problem setting

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with a Lipschitz continuous boundary $\partial\Omega$, which consists of a finite number of disjoint parts, i.e., $\partial\Omega = \Gamma_D \cup \Gamma_N \cup_{i=1}^n \Gamma_i$.

We consider the following BVP:

$$\nabla \cdot (-\mathbf{A}_{dif} \nabla u - \mathbf{a}_{con} u) + a_{sou} u = f \quad \text{in } \Omega \quad (3a)$$

$$u = g_{Dir} \quad \text{on } \Gamma_D \quad (3b)$$

$$(-\mathbf{A}_{dif} \nabla u - \mathbf{a}_{con} u) \cdot \boldsymbol{\nu} - g_{Rob} u = g_{Neu} \quad \text{on } \Gamma_N \quad (3c)$$

$$(-\mathbf{A}_{dif} \nabla u - \mathbf{a}_{con} u) \cdot \boldsymbol{\nu} = q_i + c_i \quad \text{on } \Gamma_i \quad (3d)$$

$$\int_{\Gamma_i} u = U_i \quad (3e)$$

for a given set of values $U_i \in \mathbb{R}$ and a given set of shape functions q_i , ($i = 1, \dots, n$). Here, the normal component of the flux \mathbf{q} is prescribed on Γ_i by the shape function q_i , up to an additive *unknown* real constant c_i , which has to be determined as a part of the problem.

Consider the space

$$V = \{\varphi \in W^{1,2}(\Omega) \mid \varphi = 0 \text{ on } \Gamma_D\},$$

where $W^{1,2}(\Omega)$ stands for the standard first-order Sobolev space on Ω , endowed with the $\|\cdot\|_{1,2,\Omega}$ -norm. We define the bilinear form a on $V \times V$ by

$$a(u, v) = \int_{\Omega} [\mathbf{A}_{dif} \nabla u + \mathbf{a}_{con} u] \cdot \nabla v + \int_{\Omega} a_{sou} uv + \int_{\Gamma_N} g_{Rob} uv.$$

We assume the form a to be V -elliptic, i.e.,

$$\exists C_0 > 0 : a(w, w) \geq C_0 \|w\|_{1,2,\Omega}^2, \quad \forall w \in V. \quad (4)$$

For a set of conditions on the data functions \mathbf{A}_{dif} , \mathbf{a}_{con} , a_{sou} and g_{Rob} , entering (3), which ensure the V -ellipticity of a , we may refer to [1, p.1227 and p.1245]. In addition, the functions f, g_{Dir}, g_{Neu} and q_i obey the conditions

$$f \in L_2(\Omega), g_{Dir} \in L_2(\Gamma_D), g_{Neu} \in L_2(\Gamma_N) \text{ and } q_i \in L_2(\Gamma_i), i = 1, \dots, n. \quad (5)$$

3 Uniqueness and existence

First, we prove the uniqueness of the solution of the BVP (3). In turn, this uniqueness will be used in the proof of the existence theorem.

Theorem 1 (uniqueness) *There exists at most one solution of the BVP (3).*

Proof: Assume that there exist two solutions u and w of (3). Then, we subtract the corresponding PDEs, we multiply the result by $u - w$ and we integrate the equality obtained over Ω . Applying Green's theorem, we have

$$\int_{\Omega} [\mathbf{A}_{dif} \nabla(u - w) + \mathbf{a}_{con}(u - w)] \cdot \nabla(u - w) + \int_{\Omega} a_{sou}(u - w)^2 - \int_{\partial\Omega} [\mathbf{A}_{dif} \nabla(u - w) + \mathbf{a}_{con}(u - w)] \cdot \boldsymbol{\nu}(u - w) = 0. \quad (6)$$

The integral over $\partial\Omega$ can be simplified as follows. As $u - w = 0$ on Γ_D and as moreover from (3d) the normal component of the flux of $u - w$ on Γ_i is

constant, we have

$$\begin{aligned} & \int_{\partial\Omega} [\mathbf{A}_{dif} \nabla(u-w) + \mathbf{a}_{con}(u-w)] \cdot \boldsymbol{\nu}(u-w) \\ = & - \int_{\Gamma_N} g_{Rob}(u-w)^2 + \sum_{i=1}^n [\mathbf{A}_{dif} \nabla(u-w) + \mathbf{a}_{con}(u-w)] \cdot \boldsymbol{\nu} \int_{\Gamma_i}(u-w) \\ & = - \int_{\Gamma_N} g_{Rob}(u-w)^2, \end{aligned}$$

where in the last equality we have invoked the nonlocal BC (3e). Consequently, (6) reduces to

$$a(u-w, u-w) = 0,$$

from which we conclude that $u = w$ on account of the V -ellipticity (4). ♠

We set up a constructive proof of the existence of a solution of the BVP (3). The ideas underlying the construction of the solution are as follows.

At each boundary part Γ_i , ($i = 1, \dots, n$), two conditions are imposed, one of them having a nonlocal character, the other containing an unknown parameter. When we omit the nonlocal (integral) side condition at each Γ_i and we suppose the constants c_i to be given, the resulting BVP is well-posed. We investigate the dependence of the corresponding solution on the constants c_i , ($i = 1, \dots, n$). This idea will be combined with the principle of linear superposition to find a set (c_1, \dots, c_n) for which also the nonlocal BCs at Γ_i hold, ($i = 1, \dots, n$). Thus, we consider two types of auxiliary problems.

First, we take into account the source function f , the BCs on the Dirichlet and the Neumann part of $\partial\Omega$ and the shape functions Γ_i , $i = 1, \dots, n$:

$$\begin{cases} \nabla \cdot (-\mathbf{A}_{dif} \nabla v - \mathbf{a}_{con} v) + a_{sou} v = f & \text{in } \Omega \\ v = g_{Dir} & \text{on } \Gamma_D \\ (-\mathbf{A}_{dif} \nabla v - \mathbf{a}_{con} v) \cdot \boldsymbol{\nu} - g_{Rob} v = g_{Neu} & \text{on } \Gamma_N \\ (-\mathbf{A}_{dif} \nabla v - \mathbf{a}_{con} v) \cdot \boldsymbol{\nu} = q_i & \text{on } \Gamma_i, i = 1, \dots, n. \end{cases} \quad (7)$$

Next, for $i = 1, \dots, n$, we introduce a function z_i by:

$$\begin{cases} \nabla \cdot (-\mathbf{A}_{dif} \nabla z_i - \mathbf{a}_{con} z_i) + a_{sou} z_i = 0 & \text{in } \Omega \\ z_i = 0 & \text{on } \Gamma_D \\ (-\mathbf{A}_{dif} \nabla z_i - \mathbf{a}_{con} z_i) \cdot \boldsymbol{\nu} - g_{Rob} z_i = 0 & \text{on } \Gamma_N \\ (-\mathbf{A}_{dif} \nabla z_i - \mathbf{a}_{con} z_i) \cdot \boldsymbol{\nu} = 1 & \text{on } \Gamma_i \\ (-\mathbf{A}_{dif} \nabla z_i - \mathbf{a}_{con} z_i) \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_j, j \neq i. \end{cases} \quad (8)$$

Each of these auxiliary problems has a unique solution (see, e.g., [3]).

Now, for any $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, we define the function $u_{\boldsymbol{\alpha}}$ as

$$u_{\boldsymbol{\alpha}} = v + \sum_{i=1}^n \alpha_i z_i. \quad (9)$$

By the linearity of the BVPs (7)–(8), one easily gets that $u_{\boldsymbol{\alpha}}$, (9), solves the following problem:

$$\begin{cases} \nabla \cdot (-\mathbf{A}_{dif} \nabla u_{\boldsymbol{\alpha}} - \mathbf{a}_{con} u_{\boldsymbol{\alpha}}) + a_{sou} u_{\boldsymbol{\alpha}} = f & \text{in } \Omega \\ u_{\boldsymbol{\alpha}} = g_{Dir} & \text{on } \Gamma_D \\ (-\mathbf{A}_{dif} \nabla u_{\boldsymbol{\alpha}} - \mathbf{a}_{con} u_{\boldsymbol{\alpha}}) \cdot \boldsymbol{\nu} - g_{Rob} u_{\boldsymbol{\alpha}} = g_{Neu} & \text{on } \Gamma_N \\ (-\mathbf{A}_{dif} \nabla u_{\boldsymbol{\alpha}} - \mathbf{a}_{con} u_{\boldsymbol{\alpha}}) \cdot \boldsymbol{\nu} = q_i + \alpha_i & \text{on } \Gamma_i, i = 1, \dots, n. \end{cases} \quad (10)$$

Next, we aim at determining the appropriate value of the parameter α , for which the solution u_α of (10) also satisfies the n side conditions

$$P_i(u_\alpha) \equiv \int_{\Gamma_i} u_\alpha = U_i, \quad i = 1, \dots, n. \quad (11)$$

By the linearity of the integral operator P_i , we can write

$$P_i(u_\alpha) = P_i(v) + \sum_{j=1}^n \alpha_j P_i(z_j).$$

Hence, the conditions (11) give rise to the following linear algebraic system:

$$\begin{pmatrix} P_1(z_1) & \dots & P_1(z_n) \\ \vdots & \ddots & \vdots \\ P_n(z_1) & \dots & P_n(z_n) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} U_1 - P_1(v) \\ \vdots \\ U_n - P_n(v) \end{pmatrix}. \quad (12)$$

Now, we can prove the existence of a solution of the BVP (3).

Theorem 2 *There exists a solution of the BVP (3).*

Proof: We only need to show that the matrix $\mathbf{P} = (P_i(z_j))_{i,j}$ of the linear algebraic system (12) is regular. To this end, assume that it is singular. Then at least one of its columns is a linear combination of the others. Without

loss of generality we may suppose that it concerns the last column, i.e., that there exists a set of coefficients $\lambda_1, \dots, \lambda_{n-1} \in \mathbb{R}$ such that

$$P_j \left(z_n - \sum_{i=1}^{n-1} \lambda_i z_i \right) = 0, \quad \text{for } j = 1, \dots, n. \quad (13)$$

Introduce the function $w = z_n - \sum_{i=1}^{n-1} \lambda_i z_i$. On account of (8) and (13), this function is a solution of the following BVP:

$$\left. \begin{aligned} \nabla \cdot (-\mathbf{A}_{dif} \nabla w - \mathbf{a}_{con} w) + a_{sou} w &= 0 && \text{in } \Omega \\ w &= 0 && \text{on } \Gamma_D \\ (-\mathbf{A}_{dif} \nabla w - \mathbf{a}_{con} w) \cdot \boldsymbol{\nu} - g_{Rob} w &= 0 && \text{on } \Gamma_N \\ (-\mathbf{A}_{dif} \nabla w - \mathbf{a}_{con} w) \cdot \boldsymbol{\nu} &= c_i \text{ (const)} && \text{on } \Gamma_i, \quad i = 1, \dots, n \\ P_i(w) &= 0 && i = 1, \dots, n. \end{aligned} \right\}$$

Clearly, this problem has the zero function as a trivial solution. However, by Theorem 1, this trivial solution is the only one. We conclude that

$$z_n = \sum_{i=1}^{n-1} \lambda_i z_i. \quad (14)$$

We recall that the normal components of the flux of z_i vanish on Γ_n , when $i \neq n$, see (8). Thus, (14) implies

$$(-\mathbf{A}_{dif} \nabla z_n - \mathbf{a}_{con} z_n) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \Gamma_n,$$

which contradicts the BC on Γ_n , imposed on z_n , see (8). Consequently, the matrix \mathbf{P} is regular. The unique solution of (3) is then defined by the relation (9), where $\boldsymbol{\alpha}$ is the unique solution of the algebraic system (12). ♠

We emphasize that the proof of Theorem 2 has a constructive character. First, one has to solve the well-posed auxiliary problems (7)–(8) with standard (i.e., pointwise) BCs. The desired solution will be an appropriate linear combination of the solutions of these auxiliary BVPS. The weights $\alpha_1, \dots, \alpha_n$ of this linear combination are determined as the solution of a linear algebraic system. Moreover, the values $\alpha_1, \dots, \alpha_n$ obtained determine the values of the normal component of the flux at the respective boundaries $\Gamma_1, \dots, \Gamma_n$. Of course, also the initially unknown traces of the solution of (3) on $\Gamma_1, \dots, \Gamma_n$, can now be evaluated from the constructed function $u_{\boldsymbol{\alpha}}$.

4 Numerical experiments

4.1 First example

Consider the geometrical situation shown in Figure 1a, corresponding, physically, to some soil venting facility, as mentioned in the introduction. The domain Ω is a circle with radius 2, which has four circular holes with radius 0.2. Here, Γ_P represents a passive well (an opening to the atmosphere), while Γ_1, Γ_2 and Γ_3 represent active wells (pumps). Both the domain and the pas-

sive well are centred at the origin $(0, 0)$, while Γ_1 , Γ_2 and Γ_3 are centred at $(1.5, 0)$, $(-1.2, 0.5)$ and $(0, -1.)$, respectively.

We consider the following problem:

$$\left\{ \begin{array}{ll} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_P \\ -\nabla u \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_N \\ -\nabla u \cdot \boldsymbol{\nu} = c_i \text{ (unknown constant)} & \text{on } \Gamma_i, i = 1, 2, 3 \\ \int_{\Gamma_i} u = U_i & i = 1, 2, 3, \end{array} \right. \quad (15)$$

with $U_1 = -0.1$, $U_2 = -0.2$ and $U_3 = -0.15$.

We apply the method of Section 3. As the BVP (7) is now completely homogeneous, we have $v = 0$. For the approximation of the auxiliary problems (8), we set up a finite element method with continuous, piecewise linear polynomial functions on a triangular mesh on Ω , consisting of 87552 triangles.

The resulting linear algebraic system of the type (12) has the solution

$$\alpha_1 = 0.1565868721, \quad \alpha_2 = 0.3666216670, \quad \alpha_3 = 0.2920042558. \quad (16)$$

The contour lines for the corresponding solution $u_{\boldsymbol{\alpha}}$, cf. (9), are shown in Figure 1b.

To assess the accuracy of the method, we have—conversely—considered

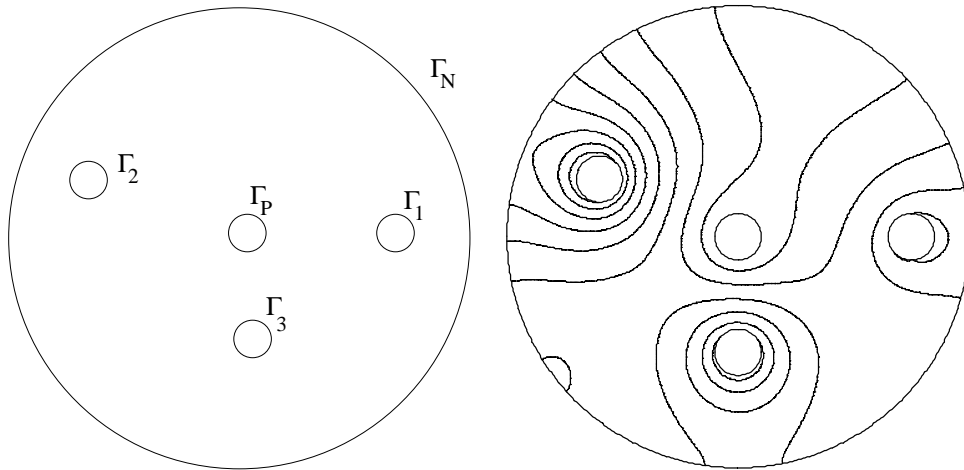


FIGURE 1: (a) The domain Ω and its boundary parts; (b) contour lines for u_α

the problem

$$\left\{ \begin{array}{ll} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_P \\ -\nabla u \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_N \\ -\nabla u \cdot \boldsymbol{\nu} = \alpha_i & \text{on } \Gamma_i \quad (i = 1, 2, 3), \end{array} \right.$$

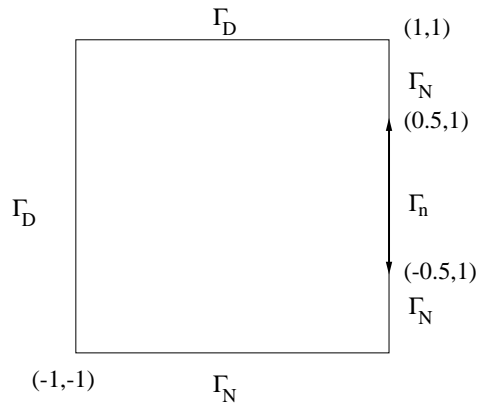
the values of α_i ($i = 1, 2, 3$) being given by (16). This classical BVP is solved by the FEM mentioned above. When evaluating the line integrals over Γ_i for the corresponding approximate solution, the values U_i , ($i = 1, 2, 3$), imposed in (15), are recovered *exactly*.

In general, because of the proposed solution strategy, the error committed using FEMs for this type of problem will be of the same order as the one for a BVP with classical BCs.

4.2 Second example

Let $\Omega = (-1, 1) \times (-1, 1)$. The boundary $\partial\Omega$ is split into three parts Γ_D , Γ_N and Γ_n , as shown in Figure 2. We consider the following BVP

$$\left\{ \begin{array}{ll} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -\nabla u \cdot \boldsymbol{\nu} = 0 & \text{on } \Gamma_N \\ -\nabla u \cdot \boldsymbol{\nu} = q + \text{const} & \text{on } \Gamma_n \\ \int_{\Gamma_n} u = -0.2 & \text{on } \Gamma_n. \end{array} \right.$$

FIGURE 2: The domain Ω and its boundary parts

The shape function q either vanishes or takes the form $q(x, y) = 1 + y$. We compute the contour lines in both cases, in order to illustrate the influence of the shape function on the flow field in the vicinity of Γ_n . The geometry and the choice of the BCs may correspond to a vertical cross-section of an air pumping well in the soil. Such a rectangular domain is normally insulated from the bottom by an impervious layer, while it is open to the atmosphere at the top side. At a sufficiently large distance from the well (left side), one may assume a Dirichlet BC. The BCs on Γ_n describe the inflow into the well, while the BCs on Γ_N (right side) reflect some symmetry conditions.

We apply the method of Section 3. The auxiliary problems (7) and (8) are solved by a FEM with continuous, piecewise linear polynomials on an unstructured, triangular mesh with 1776 triangles. The contour lines for the solution are shown in Figure 3, for both cases considered. Although the average of the solutions on Γ_n is the same, their behaviour is different, due to the shape function q , which prescribes the outflow on Γ_n .

Acknowledgements: This work was supported by the VEO-project no. 011 VO 697. The authors thank Roger Van Keer, coordinator of this project, for stimulating discussions and for his critical reading of the manuscript.

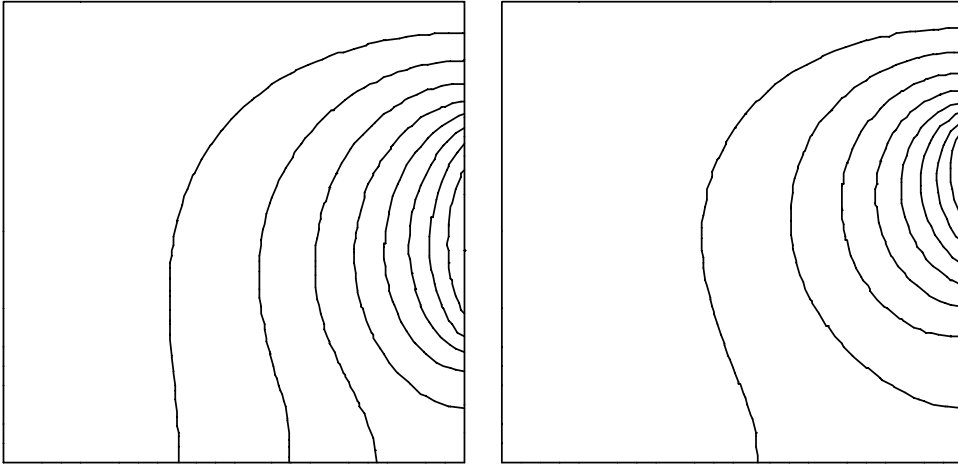


FIGURE 3: Contour lines for u_{α} : (a) for $q(x, y) = 0$; (b) for $q(x, y) = 1 + y$.

References

- [1] M. Authier *et al.* *Problèmes variationnels linéaires. Régularité.* In: R. Dautray and J.L. Lions, editors, *Analyse mathématique et calcul numérique pour les sciences et les techniques, tome 1.* Masson, Paris, 1985. [C523](#)
- [2] H. Gerke *et al.* *Optimal Control of Soil Venting: Mathematical Modeling and Applications*, Vol.127, *ISNM.* Birkhäuser, 1999. [C520](#), [C520](#)
- [3] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin, Heidelberg, 1983. [C525](#)
- [4] R. Van Keer *et al.* Computational methods for the evaluation of the electromagnetic losses in electric machinery. *Arch. Comp. Meth. Engrg.*, 5(4):385–443, 1999. [C521](#)