A stabilised mixed finite element method for thin plate splines based on biorthogonal systems

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Abstract

We propose a novel stabilised mixed finite element method for the discretisation of thin plate splines. The mixed formulation is obtained by introducing the gradient of the smoother as an additional unknown. Working with a pair of bases for the gradient of the smoother and the Lagrange multiplier, which forms a biorthogonal system, we eliminate these two variables (gradient of the smoother and Lagrange multiplier) leading to a positive definite formulation. We prove a sub-optimal a priori error estimate for the proposed finite element scheme.

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Keywords: thin plate splines, scattered data smoothing, finite element methods, saddle point problem, biorthogonal system, a priori estimate

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1 Introduction

We propose a new finite element approach for the discretisation of the thin plate spline [7, 16], which is one of the most popular approaches in scattered data fitting. Scattered data fitting problems occur in many applications such as data mining, reconstruction of geometric models, image processing, parameter estimation and optic flow [1, 8, 17].

Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2, 3\}$ be a closed and bounded region with polygonal or polyhedral boundary. We use standard notation for the norm and semi-norm of Sobolev spaces [4]. Given a set $\mathcal{G} = \{\mathbf{x}_i\}_{i=1}^N$ of scattered points in Ω, and a function r on $\mathcal{G}$ with $z_i = r(\mathbf{x}_i)$ for $i = 1, \ldots, N$, the thin plate spline is a smooth function $u \in H^2(\Omega)$ [7, 16] such that

$$u = \arg\min_{u \in H^2(\Omega)} \left(\sum_{i=1}^{N} [u(\mathbf{x}_i) - z_i]^2 + \alpha \int_\Omega \sum_{|\nu|=2} \binom{2}{\nu} (D^\nu u)^2 \, d\mathbf{x} \right) , \qquad (1)$$

where $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ is a multi-index, $|\nu| = \sum_{i=1}^d \nu_i$, and α is a positive constant. Note that $H^1(\Omega) = \{u \in L^2(\Omega), \nabla u \in [L^2(\Omega)]^d\}$, and $H^2(\Omega) = \{u \in H^1(\Omega), \nabla u \in [H^1(\Omega)]^d\}$. A conventional approach is to use radial basis functions to approximate the space $H^2(\Omega)$ in (1), which leads to a dense system matrix. The solution of such a system is very expensive when a large data set has to be modelled. In this article we propose an efficient discretisation technique for the minimisation of the functional (1). The basic idea of a finite element method is to replace the continuous space $H^2(\Omega)$ by a discrete one. To discretise the minimisation problem using a conforming approach, we need to construct a discrete finite element space which is a subset of the Sobolev space $H^2(\Omega)$. Construction of such a finite element space is expensive and difficult [6, 4]. The class of standard non-conforming finite elements [6, 4] provides a more efficient discretisation than the conforming approach. However, their implementation requires a complicated data structure, and a suitably constructed mixed formulation provides a more efficient and flexible discretisation than the non-conforming approach. Here we follow an approach used previously [9, 5, 12] to modify the original minimisation problem (1) so that the minimisation is done over the Sobolev space $H^1(\Omega)$ rather than over the Sobolev space $H^2(\Omega)$. We also aim for an efficient mixed finite element discretisation.

The rest of the article is organised as follows. In the remainder of this section, we fix some notation and introduce an alternative equivalent variational problem. Section 2 introduces a finite element solution of the problem. We recast the problem as a saddle point problem. The algebraic system motivates the usage of a pair of finite element bases (for the gradient of the smoother and the Lagrange multiplier) which forms a biorthogonal system. Section 3 is devoted to the analysis of the discrete problem. Eliminating the gradient and the Lagrange multiplier, we get a positive definite formulation of the saddle point problem for which we prove the existence of a unique solution. The final part of Section 3 shows the (sub-optimal) convergence of our finite element solution to the continuous solution.

Let the Sobolev space $H^1(\Omega) \times [H^1(\Omega)]^d$ be denoted by $\mathcal{V}$, and for two matrix-valued functions $\boldsymbol{\alpha} : \Omega \to \mathbb{R}^{d\times d}$ and $\boldsymbol{\beta} : \Omega \to \mathbb{R}^{d\times d}$, the Sobolev inner product is

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_{H^k(\Omega)} := \sum_{i=1}^{d} \sum_{j=1}^{d} (\alpha_{ij}, \beta_{ij})_{H^k(\Omega)} ,$$

where $(\boldsymbol{\alpha})_{ij} = \alpha_{ij}$ and $(\boldsymbol{\beta})_{ij} = \beta_{ij}$ with $\alpha_{ij}, \beta_{ij} \in H^k(\Omega)$. The Sobolev norm $\| \cdot \|_{H^k(\Omega)}$ is induced from Sobolev inner product. For $k = 0$, an equivalent notation,

$$(\boldsymbol{\alpha}, \boldsymbol{\beta})_{L^2(\Omega)} := \sum_{i=1}^{d} \sum_{j=1}^{d} \int_{\Omega} \alpha_{ij} \beta_{ij} \, dx = \int_{\Omega} \boldsymbol{\alpha} : \boldsymbol{\beta} \, dx ,$$

for the L^2-inner product is used and the L^2-norm $\| \cdot \|_{L^2(\Omega)}$ is induced by this inner product. We note that $\boldsymbol{\alpha} : \boldsymbol{\beta} = \sum_{i=1}^{d} \sum_{j=1}^{d} \alpha_{ij} \beta_{ij}$.

A new formulation of the minimisation problem (1) is obtained by introducing an auxiliary variable $\boldsymbol{\sigma} = \nabla u$ [9, 5],

$$[u, \boldsymbol{\sigma}] = \mathop{\arg\min}_{\substack{(v,\boldsymbol{\tau}) \in \mathcal{V} \\ \boldsymbol{\tau} = \nabla v}} \left(\sum_{i=1}^{N} [v(\mathbf{x}_i) - z_i]^2 + \alpha \| \nabla \boldsymbol{\tau} \|_{L^2(\Omega)}^2 \right) . \tag{2}$$

A finite element or discrete formulation is obtained by replacing the infinite dimensional space $\mathcal{V}$ by a finite dimensional space $\mathcal{V}_h \subset \mathcal{V}$ (also called a finite element space). The space $\mathcal{V}_h$ should be chosen carefully to guarantee convergence and efficiency of the approach.

2 Finite element problem

Let $\mathcal{T}_h$ be a quasi-uniform partition of the domain Ω in triangles or tetrahedra with mesh-size h. Let $\hat{T}$ be a reference triangle defined as

$$\hat{T} := \{(x, y) : 0 < x \, , 0 < y \, , x + y < 1\} ,$$

or a reference tetrahedron defined as

$$\hat{T} := \{(x, y, z) : 0 < x\,, 0 < y\,, 0 < z\,, x + y + z < 1\}.$$

First, we define linear and quadratic finite element spaces:

$$\mathcal{L}_h := \left\{ v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1(T)\,, T \in \mathcal{T}_h \right\}\,, \tag{3}$$

and

$$\mathcal{Q}_h := \left\{ v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_2(T)\,, T \in \mathcal{T}_h \right\}\,, \tag{4}$$

where $\mathcal{P}_n(T)$ is the polynomial space of degree $n \in \mathbb{N}$ in T [6, 4].

To obtain the discrete form of the minimisation problem (2), we introduce a finite element space $\mathcal{V}_h \subset \mathcal{V}$ defined as $\mathcal{V}_h = \mathcal{Q}_h \times [\mathcal{L}_h]^d$, and a piecewise polynomial space $\mathcal{M}_h \subset L^2(\Omega)$ based on $\mathcal{T}_h$ satisfying $\dim \mathcal{M}_h = \dim \mathcal{L}_h$. We assume the following.

Assumption 1. *There is a constant $\beta > 0$ independent of the triangulation $\mathcal{T}_h$ such that*

$$\|\phi_h\|_{L^2(\Omega)} \leqslant \beta \sup_{\mu_h \in \mathcal{M}_h \setminus \{0\}} \frac{\int_\Omega \mu_h \phi_h \, \mathbf{dx}}{\|\mu_h\|_{L^2(\Omega)}}\,, \quad \phi_h \in \mathcal{L}_h\,. \tag{5}$$

Assumption 2. *The space $\mathcal{M}_h$ has the approximation property:*

$$\inf_{\lambda_h \in \mathcal{M}_h} \|\phi - \lambda_h\|_{L^2(\Omega)} \leqslant Ch|\phi|_{H^1(\Omega)}\,, \quad \phi \in H^1(\Omega)\,. \tag{6}$$

As an example, $\mathcal{M}_h = \mathcal{L}_h \subset H^1(\Omega)$.

We utilise the greater flexibility of $\mathcal{M}_h \subset L^2(\Omega)$ to obtain an efficient finite element scheme. We replace the space $\mathcal{V}$ in (2) by our discrete space $\mathcal{V}_h$, to get our discrete problem:

Problem 3. Determine $(u_h, \boldsymbol{\sigma}_h) \in \mathcal{V}_h$ to satisfy

$$\underset{(u_h,\boldsymbol{\sigma}_h)\in\mathcal{V}_h}{\arg\min} \left(\sum_{i=1}^{N} [u_h(\mathbf{x}_i) - z_i]^2 + \alpha \|\nabla \boldsymbol{\sigma}_h\|_{L^2(\Omega)}^2 \right), \tag{7}$$

subject to

$$\langle \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h \rangle_{L^2(\Omega)} = \langle \nabla u_h, \boldsymbol{\tau}_h \rangle_{L^2(\Omega)}, \quad \boldsymbol{\tau}_h \in [\mathcal{M}_h]^d. \tag{8}$$

Replacing the constraint (8) by

$$\langle \nabla u_h, \nabla v_h \rangle_{L^2(\Omega)} = \langle \boldsymbol{\sigma}_h, \nabla v_h \rangle_{L^2(\Omega)}, \quad v_h \in \mathcal{L}_h,$$

we obtain the finite element thin plate spline presented by Roberts et al. [14], which has two drawbacks. The first is the difficult to solve saddle point structure of the system matrix arising from the discretisation. The second drawback is that it does not necessarily converge to the continuous solution of (1), although it has similar smoothing properties to the standard thin plate spline [14]. A new finite element approach was presented by Lamichhane et al. [13] to discretise the thin plate spline using bubble functions, but there is no convergence proof of this approach. In contrast, we do not use bubble functions but use a stabilised formulation leading to a true approximation of the standard thin plate spline, which converges to the exact solution of (1) when the mesh-size approaches zero.

Here our interest is to eliminate the degrees of freedom corresponding to $\boldsymbol{\sigma}_h$ and $\boldsymbol{\phi}_h$ and arrive at a formulation only depending on u_h. This dramatically reduces the size of the system matrix, and reduces it to a positive definite matrix. It is well-known that efficient numerical techniques are available to solve the positive definite system.

We start with eliminating the gradient of the smoother $\boldsymbol{\sigma}_h$ from Problem 3 and recast it as an unconstrained optimisation problem. To this end, we introduce a projection operator $R_h : L^2(\Omega) \to \mathcal{L}_h$, which is defined as

$$\int_\Omega R_h v \mu_h \, d\mathbf{x} = \int_\Omega v \mu_h \, d\mathbf{x}, v \in L^2(\Omega), \quad \mu_h \in \mathcal{M}_h.$$

The definition of R_h allows us to write the weak gradient as

$$\boldsymbol{\sigma}_h = R_h \nabla u_h ,$$

where the operator R_h is applied to the vector ∇u_h componentwise. We see that R_h is well-defined due to Assumption 1. Furthermore, the restriction of R_h to $\mathcal{L}_h$ is the identity. Hence R_h is a projection onto the space $\mathcal{L}_h$. We note that R_h is not the orthogonal projection onto $\mathcal{L}_h$ but an oblique projection onto $\mathcal{L}_h$ [15]. The operator R_h is used extensively in the context of mortar finite elements [2, 11]. Utilising R_h and denoting the vector of function values of $u \in C^0(\Omega)$ at the measurement points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ by $Pu \in \mathbb{R}^N$,

$$Pu = (u(\mathbf{x}_1), u(\mathbf{x}_2), \ldots, u(\mathbf{x}_N))^T ,$$

the minimisation Problem 3 is

$$u_h = \underset{v_h \in \mathcal{Q}_h}{\arg\min} \left(\|Pv_h\|^2 + \alpha \|\nabla (R_h \nabla v_h)\|^2_{L^2(\Omega)} - 2(Pv_h)^T \mathbf{z} \right) , \tag{9}$$

where $\mathbf{z} \in \mathbb{R}^N$. The main difficulty of this approach is that the operator R_h is not a coercive operator in the L^2-norm. There exists a function $v_h \in \mathcal{Q}_h$ with $\|\nabla v_h\|_{L^2(\Omega)} > 0$ such that $R_h(\nabla v_h) = 0$. Hence the problem is not uniquely solvable. To gain the coercivity we add a stabilisation term in the minimisation problem (9) so that our stabilised problem is to find u_h which satisfies

$$J_\alpha(u_h) = \min_{v_h \in \mathcal{Q}_h} J_\alpha(v_h) , \tag{10}$$

where

$$J_\alpha(v_h) = \|Pv_h\|^2 + \alpha \|\nabla (R_h \nabla v_h)\|^2_{L^2(\Omega)} + \|R_h \nabla v_h - \nabla v_h\|^2_{L^2(\Omega)} - 2(Pv_h)^T \mathbf{z} .$$

To show that this problem has a unique solution, we define a P-inner product $\langle \cdot , \cdot \rangle_P$ with

$$\langle u_h, v_h \rangle_P = (Pu_h)^T P v_h + \alpha \int_\Omega \nabla \boldsymbol{\sigma}_h : \nabla \boldsymbol{\tau}_h \, d\mathbf{x} + \int_\Omega (\boldsymbol{\sigma}_h - \nabla u_h) \cdot (\boldsymbol{\tau}_h - \nabla v_h) \, d\mathbf{x} ,$$

where $\boldsymbol{\sigma}_h = R_h \nabla u_h$ and $\boldsymbol{\tau}_h = R_h \nabla v_h$. It follows that

$$J_\alpha(v_h) = \langle v_h, v_h \rangle_P - 2(P v_h)^T \mathbf{z}\,.$$

The following theorem shows that the P-inner product defines an inner product on the vector space $\mathcal{Q}_h$.

Theorem 4. *Let $\alpha > 0$ and $\mathcal{G} \subset \bar{\Omega}$ have at least three non-collinear points for $d = 2$ and and four non-coplanar points for $d = 3$, then the P-inner product defined above is an inner product on the vector space $\mathcal{Q}_h$.*

Proof: In order to show that the P-inner product is indeed an inner product, we have to prove the following properties of P-inner product:

- $\langle v_h, v_h \rangle_P \geqslant 0$ and $\langle v_h, v_h \rangle_P = 0$ if and only if $v_h = 0$ and $v_h \in \mathcal{L}_h$;
- $\langle v_h + w_h, z_h \rangle_P = \langle v_h, z_h \rangle_P + \langle w_h, z_h \rangle_P$ for $v_h, w_h, z_h \in \mathcal{L}_h$;
- $\langle v_h, b z_h \rangle_P = b \langle v_h, z_h \rangle_P$ for $v_h \in \mathcal{L}_h, b \in \mathbb{R}$;
- $\langle v_h, w_h \rangle_P = \langle w_h, v_h \rangle_P$ for $v_h, w_h \in \mathcal{L}_h$.

It is trivial to show that the P-inner product satisfies the second, third and fourth properties. It is also obvious that $\langle v_h, v_h \rangle_P \geqslant 0$, and $\langle v_h, v_h \rangle_P = 0$ if $v_h = 0$. It remains to be shown that $\langle v_h, v_h \rangle_P = 0$ implies $v_h = 0$.

We have $\langle v_h, v_h \rangle_P = \|P v_h\|^2 + \alpha \|\nabla \boldsymbol{\tau}_h\|^2_{L^2(\Omega)} + \|\boldsymbol{\tau}_h - \nabla v_h\|_{L^2(\Omega)}$ with $\boldsymbol{\tau}_h = R_h \nabla v_h$. Let $\langle v_h, v_h \rangle_P = 0$, then $\|P v_h\|^2 = 0$, $\|\nabla \boldsymbol{\tau}_h\|^2_{L^2(\Omega)} = 0$ and $\|\boldsymbol{\tau}_h - \nabla v_h\|_{L^2(\Omega)} = 0$, as they are all positive. Since $\boldsymbol{\tau}_h$ is continuous, $\|\nabla \boldsymbol{\tau}_h\|_{L^2(\Omega)} = 0$ if and only if $\boldsymbol{\tau}_h$ is a constant vector function in Ω. Similarly, $\|\boldsymbol{\tau}_h - \nabla v_h\|_{L^2(\Omega)} = 0$ implies that ∇v_h is also constant in Ω, and thus v_h is a global linear function in Ω. On the other hand, $\|P v_h\| = 0$ implies that v_h is zero on $\mathcal{G} \subset \bar{\Omega}$, which contains at least three non-collinear points for $d = 2$ or four non-coplanar points for $d = 3$. Hence v_h is a global linear function which is zero at three non-collinear points for $d = 2$ or four non-coplanar points for

$d = 3$, and therefore, identically vanishes in Ω. ♠

The P-norm of an element $u_h \in \mathcal{Q}_h$ induced by the inner product $\langle \cdot, \cdot \rangle_P$ is $\|u_h\|_P^2 = \|Pu_h\|^2 + \alpha \|\nabla R_h \nabla u_h\|^2_{L^2(\Omega)} + \|R_h \nabla u_h - \nabla u_h\|^2_{L^2(\Omega)}$. Since the inner product $\langle \cdot, \cdot \rangle_P$ is symmetric, the minimisation problem (10) is equivalent to the variational problem of finding $u_h \in \mathcal{Q}_h$ such that

$$\langle u_h, v_h \rangle_P = f(v_h), \quad v_h \in \mathcal{Q}_h. \tag{11}$$

As the inner product $\langle \cdot, \cdot \rangle_P$ defines a symmetric, continuous and positive-definite bilinear form (from Theorem 4), and the linear form $f(\cdot)$ is continuous with respect to the norm $\|\cdot\|_P$, a unique solution exists.

Corollary 5. *Under the assumptions of Theorem 4, the variational problem (11) admits a unique solution which depends continuously on the data.*

The corollary follows from the fact that the resulting linear system has a positive-definite matrix.

Remark 6. Using the unique solution u_h of the variational problem (11), we have a unique solution $(u_h, \boldsymbol{\sigma}_h)$ of Problem 3 with $\boldsymbol{\sigma}_h = R_h \nabla u_h$.

We now look closely at the algebraic form of the equation $\boldsymbol{\sigma}_h = R_h \nabla u_h$. Using the same notation for the finite element functions, $\boldsymbol{\sigma}_h \in [\mathcal{L}_h]^d$ and $u_h \in \mathcal{Q}_h$, and for their vector representation, we write

$$\mathtt{D}\boldsymbol{\sigma}_h - \mathtt{B}u_h = 0,$$

where $\mathtt{D}$ is the Gram matrix corresponding to the bilinear form $(\boldsymbol{\sigma}_h, \boldsymbol{\psi}_h)$ and $\mathtt{B}$ is the matrix corresponding to the bilinear form $(\nabla u_h, \boldsymbol{\psi}_h)$. The static condensation of the degree of freedom associated with $\boldsymbol{\sigma}_h$ and $\boldsymbol{\phi}_h$ is extremely easy if $\mathtt{D}$ is a diagonal invertible matrix. We aim to achieve this.

Let $\{\varphi_1, \dots, \varphi_n\}$ be the standard nodal finite element basis of $\mathcal{L}_h$. We define a space $\mathcal{M}_h$ spanned by the basis $\{\mu_1, \dots, \mu_n\}$, where the basis functions of $\mathcal{L}_h$

and $\mathcal{M}_h$ satisfy a biorthogonality condition,

$$\int_\Omega \mu_i \, \varphi_j \, dx = c_j \delta_{ij} \,, \quad c_j \neq 0 \,, \quad 1 \leqslant i, j \leqslant n \,. \tag{12}$$

Here, $n := \dim \mathcal{M}_h = \dim \mathcal{L}_h$, δ_{ij} is the Kronecker delta, and c_j a positive scaling factor. The scaling factor c_j is chosen to be proportional to the area $|\operatorname{supp} \varphi_j|$.

For the reference triangle $\hat{T} := \{(x, y) : 0 < x, 0 < y, x + y < 1\}$, the basis functions for linear finite elements in two dimensions are

$$\hat{\mu}_1 := 3 - 4x - 4y \,, \quad \hat{\mu}_2 := 4x - 1 \,, \quad \hat{\mu}_3 := 4y - 1 \,.$$

The basis functions $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\mu}_3$ are associated with the three vertices of the reference triangle, $(0, 0)$, $(1, 0)$ and $(0, 1)$. For the reference tetrahedron $\hat{T} := \{(x, y, z) : 0 < x \,, 0 < y \,, 0 < z \,, x + y + z < 1\}$, the basis functions for linear finite elements in two dimensions are

$$\hat{\mu}_1 := 4 - 5x - 5y - 5z \,, \quad \hat{\mu}_2 := 5x - 1 \,, \quad \hat{\mu}_3 := 5y - 1 \,, \quad \hat{\mu}_4 := 5z - 1 \,.$$

The basis functions $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\mu}_4$ are associated with the four vertices of the reference tetrahedron, $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

The global basis functions for the test space are constructed by glueing the local basis functions together, and thus the assembling process is exactly the same as in the standard finite element method. These global basis functions then satisfy the condition of biorthogonality (12) with global finite element basis functions, and they satisfy Assumptions 1 and 2. As these functions in $\mathcal{M}_h$ are defined in exactly the same way as the finite element basis functions in $\mathcal{L}_h$, they satisfy $\operatorname{supp} \mu_i = \operatorname{supp} \varphi_i$ for $i = 1, \ldots, n$.

3 An a priori error estimate

In this section we focus on analysing the a priori error estimate. According to the biorthogonality relation (12) between the basis functions of $\mathcal{L}_h$ and $\mathcal{M}_h$, the action of operator R_h on a function $v \in L^2(\Omega)$ is

$$R_h v = \sum_{i=1}^{n} \frac{\int_\Omega \mu_i v \, dx}{c_i} \varphi_i \,. \tag{13}$$

Consequently, the operator R_h is local in the sense described below. Let $S(T')$ be the patch of an element $T' \in \mathcal{T}_h$ which is the interior of the closed set

$$\bar{S}(T') = \overline{\bigcup \{T \in \mathcal{T}_h : \partial T \cap \partial T' \neq \emptyset\}} \,. \tag{14}$$

Then R_h is local in the sense that for any $v \in L^2(\Omega)$, the value of $R_h v$ at any point in $T \in \mathcal{T}_h$ only depends on the values of v in $S(T)$. We list the main properties of the oblique projection operator R_h in the following lemma. This lemma was proved by Kim et al. [10] and Lamichhane [11], where R_h is introduced as the mortar projection operator.

Lemma 7. *Under Assumptions 1 and 2 there exist constants* C_1, C_2, C_3 *and* C_4, *independent of mesh-size* h, *such that:*

- **Stability in L^2-norm**,

$$\|R_h v\|_{L^2(\Omega)} \leqslant C_1 \|v\|_{L^2(\Omega)} \,, \quad v \in L^2(\Omega) \,; \tag{15}$$

- **Stability in H^1-norm**,

$$|R_h v|_{H^1(\Omega)} \leqslant C_2 |v|_{H^1(\Omega)} \,, \quad v \in H^1(\Omega) \,; \tag{16}$$

- **Approximation property**, *If* $v \in H^{s+1}(\Omega)$ *with* $0 \leqslant s \leqslant 1$,

$$\begin{aligned} \|v - R_h v\|_{L^2(\Omega)} &\leqslant C_3 h^{1+s} |v|_{H^{s+1}(\Omega)} \,, \\ \|v - R_h v\|_{H^1(\Omega)} &\leqslant C_4 h^{s} |v|_{H^{s+1}(\Omega)} \,. \end{aligned} \tag{17}$$

In the following, we use a generic constant C, which takes different values at different places but is always independent of the mesh-size h. To analyse the error estimate we introduce the *energy norm*

$$\|(u, \boldsymbol{\sigma})\|_A := \sqrt{\|Pu\|^2 + \alpha |\boldsymbol{\sigma}|^2_{H^1(\Omega)} + \|\boldsymbol{\sigma} - \nabla u\|^2_{L^2(\Omega)}} \,, \tag{18}$$

where $(u, \boldsymbol{\sigma}) \in \tilde{V} \times [H^1(\Omega)]^d$ and $\tilde{V} = C^0(\Omega) \cap H^1(\Omega)$. The following theorem is important for the a priori error estimate and was proved by Ciarlet [6] and Lamichhane [12].

Theorem 8. *Assume that u is the solution of problem (1) satisfying $u \in H^4(\Omega)$, $\boldsymbol{\sigma} = \nabla u$ and $\boldsymbol{\phi} = \alpha \Delta \boldsymbol{\sigma}$, and u_h is the solution of problem (11), and $\boldsymbol{\sigma}_h = R_h \nabla u_h$. Then there exists a constant $C > 0$ independent of the mesh-size h such that*

$$\|(u - u_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_A \leqslant C \left(\inf_{(w_h, \boldsymbol{\theta}_h) \in \mathcal{K}_h} \|(u - w_h, \boldsymbol{\sigma} - \boldsymbol{\theta}_h)\|_A + h |\boldsymbol{\phi}|_{H^1(\Omega)} \right),$$

where $\mathcal{K}_h = \{(v_h, \boldsymbol{\tau}_h) \in \mathcal{V}_h | \boldsymbol{\tau}_h = R_h(\nabla v_h)\}$.

Theorem 9. *Under the assumptions of Theorem 8, there exists $(v_h, \boldsymbol{\tau}_h) \in \mathcal{K}_h$ such that*

$$\|(u - v_h, \boldsymbol{\sigma} - \boldsymbol{\tau}_h)\|_A \leqslant Ch \|u\|_{H^3(\Omega)}. \tag{19}$$

Proof: Let v_h be the piecewise quadratic Lagrange interpolant of u with respect to the mesh $\mathcal{T}_h$ and $\boldsymbol{\tau}_h = R_h \nabla v_h$. Then it is well-known that

$$\|u - v_h\|_{H^k(\Omega)} \leqslant Ch^{2-k} |u|_{H^2(\Omega)}, \quad k = 0, 1. \tag{20}$$

Moreover,

$$\|P(u - v_h)\|^2 \leqslant Ch^2 |u|^2_{H^2(\Omega)}. \tag{21}$$

Using the definition of the error in the energy norm $\|\cdot\|_A$, it is now sufficient to show that

$$\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{H^1(\Omega)} \leqslant h \|u\|_{H^3(\Omega)}.$$

Since $u \in H^3(S(T)) \cap H^1(\Omega)$ for $T \in \mathcal{T}_h$ [3],

$$\|\nabla u - R_h \nabla v_h\|_{L^2(T)} \leqslant Ch^2 \|u\|_{H^3(S(T))}.$$

Hence

$$\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{L^2(\Omega)} \leqslant Ch^2 \|u\|_{H^3(\Omega)}. \tag{22}$$

Now, using a triangle inequality, an inverse estimate and projection property of R_h,

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_{H^1(\Omega)} &\leqslant \|\boldsymbol{\sigma}-R_h\boldsymbol{\sigma}\|_{H^1(\Omega)}+\|R_h\boldsymbol{\sigma}-R_h\nabla v_h\|_{H^1(\Omega)}\\ &\leqslant C\left(\|\boldsymbol{\sigma}-R_h\boldsymbol{\sigma}\|_{H^1(\Omega)}+\frac{1}{h}\|R_h\boldsymbol{\sigma}-R_h\nabla v_h\|_{L^2(\Omega)}\right)\\ &\leqslant C\left(\|\boldsymbol{\sigma}-R_h\boldsymbol{\sigma}\|_{H^1(\Omega)}+\frac{1}{h}\|\boldsymbol{\sigma}-R_h\nabla v_h\|_{L^2(\Omega)}\right).\end{aligned}$$

The first term on the right has the correct approximation from Lemma 7, and the second term is from (22). ♠

Using the results of Theorems 8 and 9, we get the following approximation result for the discrete solution.

Corollary 10. *Assume that u is the solution of continuous problem (1) with $u \in H^4(\Omega)$, $\boldsymbol{\sigma} = \nabla u$ and $\boldsymbol{\phi} = \alpha\Delta\boldsymbol{\sigma}$, and u_h is that of discrete problem (11) with $\boldsymbol{\sigma}_h = R_h\nabla u_h$ and $\mathcal{V}_h = \mathcal{Q}_h \times [\mathcal{L}_h]^d$. Then there exists a constant $C > 0$ independent of the mesh-size h such that*

$$\|(u-u_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_A \leqslant Ch\left(\|u\|_{H^3(\Omega)}+|\boldsymbol{\phi}|_{H^1(\Omega)}\right).$$

4 Conclusion

We presented a mixed finite element approach to approximate the solution of the standard thin plate spline in two and three dimensions. This mixed finite element method introduces two additional unknowns: the gradient of the smoother and the Lagrange multiplier. Working with a system for the bases of these two additional unknowns, they can be statically condensed out of the system, leading to an efficient finite element method for the data smoothing problem. This is the main advantage of this approach. We also proved the convergence of this method. However, as in the case of the a priori error estimate of a mixed finite element for the biharmonic problem [6], the convergence to the exact solution is not optimal in the sense that one expects a quadratic order of convergence using a quadratic finite element space for the smoother. The advantage of our approach is these two additional unknowns can be eliminated by inverting a diagonal matrix leading to an efficient numerical scheme.

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