On discrete GB-splines

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Abstract

Explicit formulae and recurrence relations are obtained for discrete generalized B-splines (discrete GB-splines for short). Properties of discrete GB-splines and their series are studied. It is shown that the series of discrete GB-splines is a variation diminishing function and the systems of discrete GB-splines are weak Chebyshev systems.

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1 Introduction

The tools of generalized splines and GB-splines are widely used in solving problems of shape-preserving approximation (e.g., see [7]). Recently, in [1] a difference method for constructing shape-preserving hyperbolic tension splines as solutions of multipoint boundary value problems was developed. Such an approach permits us to avoid the computation of hyperbolic functions and has substantial other advantages. However, the extension of a mesh solution will be a discrete hyperbolic tension spline.

The contents of this paper is as follows. In Section 2 we give a definition of a discrete generalized spline. Next, we construct a minimum length local support basis (whose elements are denoted as discrete GB-splines) of the new spline; see Section 3. Properties of GB-splines are discussed in Section 4, while the local approximation by discrete GB-splines of a given continuous function from its samples is considered in Section 5. In Section 6 we derive recurrence formulae for calculations with discrete GB-splines. The properties of GB-spline series are summarized in Section 7.

2 Discrete generalized splines

Let a partition $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a, b]$ be given. We will denote by S_4^{DG} the space of continuous functions whose restriction to a subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$ is spanned by the system of four linearly independent functions $\{1, x, \Phi_i, \Psi_i\}$. In addition, we assume that each function in S_4^{DG} is smooth in the sense that for given $\tau_i^{L_j} > 0$ and $\tau_i^{R_j} > 0$, $j = i-1, i$, the values of its first and second central divided differences with respect to the points $x_i - \tau_i^{L_{i-1}}$, x_i, $x_i + \tau_i^{R_{i-1}}$ and $x_i - \tau_i^{L_i}$, x_i, $x_i + \tau_i^{R_i}$ coincide.

Given a continuous function S we introduce the difference operators

$$\begin{aligned} D_1S(x) \equiv D_{i,1}S(x) &= (\lambda_i^{R_i} S[x-\tau_i^{L_i}, x] + \lambda_i^{L_i} S[x, x+\tau_i^{R_i}])(1-t) \\ &\quad +(\lambda_{i+1}^{R_i} S[x-\tau_{i+1}^{L_i}, x] + \lambda_{i+1}^{L_i} S[x, x+\tau_{i+1}^{R_i}])t, \\ D_2S(x) \equiv D_{i,2}S(x) &= 2S[x-\tau_i^{L_i}, x, x+\tau_i^{R_i}](1-t) \\ &\quad +2S[x-\tau_{i+1}^{L_i}, x, x+\tau_{i+1}^{R_i}]t, \\ &\qquad x \in [x_i, x_{i+1}), \quad i = 0, \ldots, N-1, \end{aligned}$$

where $\lambda_j^{R_i} = 1 - \lambda_j^{L_i} = \tau_j^{R_i}/(\tau_j^{L_i} + \tau_j^{R_i})$, $j = i, i+1$ and $t = (x - x_i)/h_i$, $h_i = x_{i+1} - x_i$. The square parentheses denote the usual first and second divided differences of the function S with respect to the argument values $x_j - \tau_j^{L_i}$, x_j, $x_j + \tau_j^{R_i}$, $j = i, i+1$.

Definition 1 *A discrete generalized spline is a function $S \in S_4^{DG}$ such that*

1. *for any $x \in [x_i, x_{i+1}]$, $i = 0, \ldots, N-1$*

$$\begin{aligned} S(x) \equiv S_i(x) &= [S(x_i) - \Phi_i(x_i)M_i](1-t) \\ &\quad + [S(x_{i+1}) - \Psi_i(x_{i+1})M_{i+1}]t \\ &\quad + \Phi_i(x)M_i + \Psi_i(x)M_{i+1}, \end{aligned} \tag{1}$$

where $M_j = D_{i,2}S_i(x_j)$, $j = i, i+1$, and the functions Φ_i and Ψ_i are subject to the constraints

$$\begin{aligned} \Phi_i(x_{i+1} - \tau_{i+1}^{L_i}) &= \Phi_i(x_{i+1}) = \Phi_i(x_{i+1} + \tau_{i+1}^{R_i}) = 0, \\ \Psi_i(x_i - \tau_i^{L_i}) &= \Psi_i(x_i) = \Psi_i(x_i + \tau_i^{R_i}) = 0, \\ D_{i,2}\Phi_i(x_i) &= 1, \quad D_{i,2}\Psi_i(x_{i+1}) = 1; \end{aligned} \tag{2}$$

2. S satisfies the following smoothness conditions

$$\begin{aligned} S_{i-1}(x_i) &= S_i(x_i), \\ D_{i-1,1}S_{i-1}(x_i) &= D_{i,1}S_i(x_i), \quad i = 1, \ldots, N-1. \\ D_{i-1,2}S_{i-1}(x_i) &= D_{i,2}S_i(x_i), \end{aligned} \tag{3}$$

This definition generalizes the notion of a discrete polynomial spline in [9] and of a generalized spline in [5, 6]. The latter one can be obtained by setting $\tau_j^{L_i} = \tau_j^{R_i} = 0$, $j = i, i+1$ for all i. If $\tau_i^{L_j} = \tau_i^L$ and $\tau_i^{R_j} = \tau_i^R$, $j = i-1, i$ then according to smoothness conditions (3) the values of the functions S_{i-1} and S_i at the three consecutive points $x_i - \tau_i^L$, x_i, $x_i + \tau_i^R$ coincide. Setting $\tau_j^{L_i} = \tau_j^{R_i} = \tau_i$, $j = i, i+1$ we obtain $D_{1,i}S(x) = S[x - \tau_i, x + \tau_i]$ and $D_{2,i}S(x) = S[x - \tau_i, x, x + \tau_i]$, which is the case discussed in [1].

The functions Φ_i and Ψ_i depend on the tension parameters which influence the behaviour of S fundamentally. We call them the *defining functions*. In practice one takes $\Phi_i(x) = \Phi_i(p_i, x)$, $\Psi_i(x) = \Psi_i(q_i, x)$, $0 \le p_i, q_i < \infty$. In the limiting case when $p_i, q_i \to \infty$ we require that $\lim_{p_i \to \infty} \Phi_i(p_i, x) = 0$, $x \in (x_i, x_{i+1}]$ and $\lim_{q_i \to \infty} \Psi_i(q_i, x) = 0$, $x \in [x_i, x_{i+1})$ so that the function S in formula (1) turns into a linear function. Additionally, we require that if $p_i = q_i = 0$ for all i, then we get a discrete cubic spline. If $\tau_i^{L_j} = \tau_i^{R_j} = \tau_i$, $j = i-1, i$ for all i then this spline coincides with a discrete cubic spline of [10]. The case $\tau_i = \tau$ for all i was considered in [8].

3 Construction of discrete GB-splines

Let us construct a basis for the space of discrete generalized splines S_4^{DG} by using functions which have local supports of minimum length. Since $\dim(S_4^{DG}) = 4N - 3(N-1) = N+3$ we extend the grid Δ by adding the points x_j, $j = -3, -2, -1, N+1, N+2, N+3$, such that $x_{-3} < x_{-2} < x_{-1} < a$, $b < x_{N+1} < x_{N+2} < x_{N+3}$.

We demand that the discrete GB-splines B_i, $i = -3, \ldots, N-1$ have the properties

$$\begin{aligned} \mathrm{B}_i(x) &> 0, \quad x \in (x_i + \tau_i^{R_i}, x_{i+4} - \tau_{i+4}^{L_{i+3}}), && (4)\\ \mathrm{B}_i(x) &\equiv 0, \quad x \notin (x_i, x_{i+4}), \\ \sum_{j=-3}^{N-1} \mathrm{B}_j(x) &\equiv 1, \quad x \in [a, b]. && (5) \end{aligned}$$

According to (1), on the interval $[x_j, x_{j+1}]$, $j = i, \ldots, i+3$, the discrete GB-spline B_i has the form

$$\mathrm{B}_i(x) \equiv \mathrm{B}_{i,j}(x) = P_{i,j}(x) + \Phi_j(x) M_{j,\mathrm{B}_i} + \Psi_j(x) M_{j+1,\mathrm{B}_i}, \qquad (6)$$

where $P_{i,j}$ is a polynomial of the first degree and $M_{l,\mathrm{B}_i} = D_{j,2}\mathrm{B}_i(x_l)$, $l = j, j+1$ are constants to be determined. The smoothness conditions (3) together with the constraints (2) give the following relations

$$\begin{aligned} P_{i,j}(x_j) &= P_{i,j-1}(x_j) + z_j M_{j,\mathrm{B}_i}, \\ D_{j,1} P_{i,j}(x_j) &= D_{j-1,1} P_{i,j-1}(x_j) + c_{j-1,2} M_{j,\mathrm{B}_i}, \end{aligned}$$

where

$$\begin{aligned} z_j &\equiv z_j(x_j) = \Psi_{j-1}(x_j) - \Phi_j(x_j), \\ c_{j-1,2} &= D_{j-1,1}\Psi_{j-1}(x_j) - D_{j,1}\Phi_j(x_j). \end{aligned}$$

Thus in (6)

$$P_{i,j}(x) = P_{i,j-1}(x) + [z_j + c_{j-1,2}(x - x_j)]M_{j,\mathrm{B}_i}. \tag{7}$$

By repeated use of this formula we get

$$P_{i,j}(x) = \sum_{l=i+1}^{j} [z_l + c_{l-1,2}(x - x_l)]M_{l,\mathrm{B}_i} = -\sum_{l=j+1}^{i+3} [z_l + c_{l-1,2}(x - x_l)]M_{l,\mathrm{B}_i}.$$

As B_i vanishes outside the interval (x_i, x_{i+4}), we have from (7) that $P_{i,j} \equiv 0$ for $j = i, i+3$. In particular, the following identity is valid

$$\sum_{j=i+1}^{i+3} [z_j + c_{j-1,2}(x - x_j)]M_{j,\mathrm{B}_i} \equiv 0,$$

from which one obtains the equalities

$$\sum_{j=i+1}^{i+3} c_{j-1,2}\, y_j^r M_{j,\mathrm{B}_i} = 0, \quad r = 0, 1, \quad y_j = x_j - \frac{z_j}{c_{j-1,2}}. \tag{8}$$

Thus the formula for the discrete GB-spline B_i takes the form

$$\mathrm{B}_i(x) = \begin{cases} \Psi_i(x) M_{i+1,\mathrm{B}_i}, & x \in [x_i, x_{i+1}), \\ (x - y_{i+1}) c_{i,2} M_{i+1,\mathrm{B}_i} + \Phi_{i+1}(x) M_{i+1,\mathrm{B}_i} & \\ \quad + \Psi_{i+1}(x) M_{i+2,\mathrm{B}_i}, & x \in [x_{i+1}, x_{i+2}), \\ (y_{i+3} - x) c_{i+2,2} M_{i+3,\mathrm{B}_i} + \Phi_{i+2}(x) M_{i+2,\mathrm{B}_i} & \\ \quad + \Psi_{i+2}(x) M_{i+3,\mathrm{B}_i}, & x \in [x_{i+2}, x_{i+3}), \\ \Phi_{i+3}(x) M_{i+3,\mathrm{B}_i}, & x \in [x_{i+3}, x_{i+4}), \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Substituting formula (9) into the normalization condition (5) written for $x \in [x_i, x_{i+1}]$, we obtain

$$\begin{aligned} \sum_{j=i-3}^{i} \mathrm{B}_j(x) \quad = \quad & \Phi_i(x) \sum_{j=i-3}^{i-1} M_{i,\mathrm{B}_j} + \Psi_i(x) \sum_{j=i-2}^{i} M_{i+1,\mathrm{B}_j} \\ & + (y_{i+1} - x) c_{i,2} M_{i+1,\mathrm{B}_{i-2}} + (x - y_i) c_{i-1,2} M_{i,\mathrm{B}_{i-1}} \equiv 1. \end{aligned}$$

As according to (5)

$$\sum_{j=i-3}^{i-1} M_{i,\mathrm{B}_j} = \sum_{j=i-2}^{i} M_{i+1,\mathrm{B}_j} = 0 \tag{10}$$

the following identity is valid

$$(y_{i+1} - x) c_{i,2} M_{i+1,\mathrm{B}_{i-2}} + (x - y_i) c_{i-1,2} M_{i,\mathrm{B}_{i-1}} \equiv 1.$$

From here one gets the equalities

$$y_{i+1}^r c_{i,2} M_{i+1,\mathrm{B}_{i-2}} - y_i^r c_{i-1,2} M_{i,\mathrm{B}_{i-1}} \equiv \delta_{1,r}, \quad r = 0, 1,$$

where $\delta_{1,r}$ is the Kronecker symbol. Solving this system of equations and using (8) or (10), we obtain

$$\begin{aligned} M_{j,\mathrm{B}_i} &= \frac{y_{i+3} - y_{i+1}}{c_{j-1,2}\omega'_{i+1}(y_j)}, \quad j = i+1, i+2, i+3, \\ \omega_{i+1}(x) &= (x - y_{i+1})(x - y_{i+2})(x - y_{i+3}) \end{aligned}$$

or with the notation $c_{j,3} = y_{j+2} - y_{j+1}$, $j = i, i+1$,

$$\begin{aligned} M_{i+1,\mathrm{B}_i} &= \frac{1}{c_{i,2}c_{i,3}}, \\ M_{i+2,\mathrm{B}_i} &= -\frac{1}{c_{i+1,2}}\left(\frac{1}{c_{i,3}} + \frac{1}{c_{i+1,3}}\right), \\ M_{i+3,\mathrm{B}_i} &= \frac{1}{c_{i+2,2}c_{i+1,3}}. \end{aligned} \tag{11}$$

4 Properties of discrete GB-splines

The functions B_j, $j = -3, \ldots, N-1$ possess many of the properties inherent in usual discrete polynomial B-splines. To provide inequality (4), in what follows we need to impose additional conditions on the functions Φ_j and Ψ_j.

The proofs of the following four assertions repeat those given in [5].

Lemma 2 *If the conditions*

$$\begin{aligned} 0 &< 2h_{j-1}^{-1}\Psi_{j-1}(x_j) < D_{j-1,1}\Psi_{j-1}(x_j), \\ 0 &< 2h_j^{-1}\Phi_j(x_j) < -D_{j,1}\Phi_j(x_j), \quad j = i+1, i+2, i+3 \end{aligned} \tag{12}$$

are satisfied, then in (11) $c_{j,k} > 0$, $j = i, \dots, i+4-k$; $k = 2, 3$, *and*

$$(-1)^{j-i-1} M_{j,\mathrm{B}_i} > 0, \quad j = i+1, i+2, i+3. \tag{13}$$

Theorem 3 *Let the conditions of Lemma 2 be satisfied, the functions* Φ_j *and* Ψ_j *be convex and* $D_{j,2}\Phi_j$ *and* $D_{j,2}\Psi_j$ *be strictly monotone on the interval* $[x_j, x_{j+1}]$ *for all* j. *Then the functions* B_j, $j = -3, \dots, N-1$ *have the following properties:*

1. $\mathrm{B}_j(x) > 0$ *for* $x \in (x_j + \tau_j^{R_j}, x_{j+4} - \tau_{j+4}^{L_{j+3}})$, *and* $\mathrm{B}_j(x) \equiv 0$ *if* $x \notin (x_j, x_{j+4})$;
2. B_j *satisfies the smoothness conditions (3);*
3. $\sum_{j=-3}^{N-1} y_{j+2}^r \mathrm{B}_j(x) \equiv x^r$, $r = 0, 1$ *for* $x \in [a, b]$, $\Phi_j(x) = c_{j-1,2}c_{j-2,3}\mathrm{B}_{j-3}(x)$, $\Psi_j(x) = c_{j,2}c_{j,3}\mathrm{B}_j(x)$ *for* $x \in [x_j, x_{j+1}]$, $j = 0, \dots, N-1$.

Lemma 4 *The function* B_i *has support of minimum length.*

Theorem 5 *The functions* B_i*,* $i = -3, \ldots, N-1$*, are linearly independent and form a basis of the space* S_4^{DG} *of discrete generalized splines.*

5 Local approximation by discrete GB-splines

According to Theorem 5, any discrete generalized spline $S \in S_4^{DG}$ can be uniquely written in the form

$$S(x) = \sum_{j=-3}^{N-1} b_j \mathrm{B}_j(x) \tag{14}$$

for some constant coefficients b_j.

If the coefficients b_j in (14) are known, then by virtue of formula (9) we can write out an expression for the discrete generalized spline S on the interval $[x_i, x_{i+1}]$, which is convenient for calculations,

$$S(x) = b_{i-2} + b_{i-1}^{(1)}(x - y_i) + b_{i-1}^{(2)}\Phi_i(x) + b_i^{(2)}\Psi_i(x), \tag{15}$$

where

$$b_k^{(k)} = \frac{b_j^{(k-1)} - b_{j-1}^{(k-1)}}{c_{j,4-k}}, \quad k = 1, 2; \quad b_j^{(0)} = b_j. \tag{16}$$

The representations (14) and (15) allow us to find a simple and effective way to approximate a given continuous function f from its samples.

Theorem 6 *Let a continuous function f be given by its samples $f(y_j)$, $j = -1, \ldots, N+1$. Then for $b_j = f(y_{j+2})$, $j = -3, \ldots, N-1$, formula (14) is exact for polynomials of the first degree and provides a formula for local approximation.*

Proof: It suffices to prove that the identities

$$\sum_{j=-3}^{N-1} y_{j+2}^r \mathrm{B}_j(x) \equiv x^r, \quad r = 0, 1 \tag{17}$$

hold for $x \in [a, b]$. Using formula (15) with the coefficients $b_{j-2} = 1$ and $b_{j-2} = y_j$, $j = i-1, i, i+1, i+2$, for an arbitrary interval $[x_i, x_{i+1}]$, we find that identities (17) hold.

For $b_{j-2} = f(y_j)$, formula (15) can be rewritten as

$$\begin{aligned} S(x) \;\; = \;\; & f(y_i) + f[y_i, y_{i+1}](x - y_i) + (y_{i+1} - y_{i-1}) f[y_{i-1}, y_i, y_{i+1}] c_{i-1,2}^{-1} \Phi_i(x) \\ & + (y_{i+2} - y_i) f[y_i, y_{i+1}, y_{i+2}] c_{i,2}^{-1} \Psi_i(x), \quad x \in [x_i, x_{i+1}]. \end{aligned}$$

This is the formula of local approximation. The theorem is thus proved. ♠

Corollary 7 *Let a continuous function f be given by its samples $f_j = f(x_j)$, $j = -2, \ldots, N+2$. Then by setting*

$$b_{j-2} = f_j - \frac{1}{c_{j-1,2}}\Big(\Psi_{j-1}(x_j) f[x_j, x_{j+1}] - \Phi_j(x_j) f[x_{j-1}, x_j]\Big) \qquad (18)$$

in (14), we obtain a formula of three-point local approximation, which is exact for polynomials of the first degree.

Proof: To prove the corollary, it is sufficient to take the monomials 1 and x as f. Then according to (18), we obtain $b_{j-2} = 1$ and $b_{j-2} = y_j$ and it only remains to make use of identities (17). This proves the corollary. ♠

Equation (15) permits us to write the coefficients of the spline S in its representation (14) of the form

$$b_{j-2} = \begin{cases} S(y_j) - D_{j-1,2}S(x_{j-1})\Phi_{j-1}(y_j) - D_{j,2}S(x_j)\Psi_{j-1}(y_j), & y_j < x_j, \\ S(y_j) - D_{j,2}S(x_j)\Phi_j(y_j) - D_{j+1,2}S(x_{j+1})\Psi_j(y_j), & y_j \geq x_j. \end{cases}$$

According to this formula we have $b_{j-2} = S(y_j) + O(\overline{h}_j^2)$, $\overline{h}_j = \max(h_{j-1}, h_j)$. Hence it follows that the control polygon (e.g., see [4]) converges quadratically to the function f when $b_{j-2} = f(y_j)$, or if the formula (18) is used.

6 Recurrence formulae for discrete GB-splines

Let us define functions

$$\mathrm{B}_{j,2}(x) = \begin{cases} D_{j,2}\Psi_j(x), & x \in [x_j, x_{j+1}), \\ D_{j+1,2}\Phi_{j+1}(x), & x \in [x_{j+1}, x_{j+2}], \\ 0, & \text{otherwise}, \end{cases} \qquad j = i, i+1, i+2. \tag{19}$$

We assume that the functions $D_{j,2}\Psi_j$ and $D_{j+1,2}\Phi_{j+1}$ are strictly monotone on $[x_j, x_{j+1})$ and $[x_{j+1}, x_{j+2}]$ respectively. The splines $\mathrm{B}_{j,2}$ are a generalization of the "hat-functions" for polynomial B-splines. They are nonnegative and, furthermore, $\mathrm{B}_{j,2}(x_{j+l}) = \delta_{1,l}$, $l = 0, 1, 2$.

According to (9), (11) and (19) the function $D_2\mathrm{B}_i$ can be written as

$$\begin{aligned} D_2\mathrm{B}_i(x) &= \sum_{j=i+1}^{i+3} M_{j,\mathrm{B}_i}\mathrm{B}_{j-1,2}(x) \\ &= \frac{1}{c_{i,3}}\Big(\frac{\mathrm{B}_{i,2}(x)}{c_{i,2}} - \frac{\mathrm{B}_{i+1,2}(x)}{c_{i+1,2}}\Big) - \frac{1}{c_{i+1,3}}\Big(\frac{\mathrm{B}_{i+1,2}(x)}{c_{i+1,2}} - \frac{\mathrm{B}_{i+2,2}(x)}{c_{i+2,2}}\Big). \end{aligned} \tag{20}$$

The function $D_1\mathrm{B}_i$ satisfies the relation

$$D_1\mathrm{B}_i(x) = \frac{\mathrm{B}_{i,3}(x)}{c_{i,3}} - \frac{\mathrm{B}_{i+1,3}(x)}{c_{i+1,3}}, \tag{21}$$

where

$$\mathrm{B}_{j,3}(x)=\begin{cases}\dfrac{D_{j,1}\Psi_j(x)}{c_{j,2}}, & x\in[x_j,x_{j+1}),\\ 1+\dfrac{D_{j+1,1}\Phi_{j+1}(x)}{c_{j,2}}-\dfrac{D_{j+1,1}\Psi_{j+1}(x)}{c_{j+1,2}}, & x\in[x_{j+1},x_{j+2}),\\ -\dfrac{D_{j+2,1}\Phi_{j+2}(x)}{c_{j+1,2}}, & x\in[x_{j+2},x_{j+3}),\\ 0, & \text{otherwise.}\end{cases} \tag{22}$$

Using formula (22) it is easy to show that functions $\mathrm{B}_{j,3}$, $j=-2,\dots,N-1$ satisfy the first and second smoothness conditions in (3), have supports of minimum length, are linearly independent and form a partition of unity,

$$\sum_{j=1}^{N-1}\mathrm{B}_{j,3}(x)\equiv 1,\quad x\in[a,b].$$

Applying formulae (20) and (21) to the representation (14) we also obtain

$$D_1S(x)=\sum_{j=-2}^{N-1}b_j^{(1)}\mathrm{B}_{j,3}(x),\quad D_2S(x)=\sum_{j=-1}^{N-1}b_j^{(2)}\mathrm{B}_{j,2}(x), \tag{23}$$

where $b_j^{(k)}$, $k=1,2$ are defined in (16).

7 Series of discrete GB-splines (uniform case)

Let us suppose that each step size $h_i = x_{i+1} - x_i$ of the mesh $\Delta : a = x_0 < x_1 < \cdots < x_N = b$ is an integer multiple of the same tabulation step, τ, of some detailed uniform refinement on $[a, b]$.

For $\theta \in \mathbb{R}$, $\tau > 0$ define

$$\mathbb{R}_{\theta\tau} = \{\theta + i\tau \mid i \text{ is an integer}\}$$

and let $\mathbb{R}_{\theta 0} = \mathbb{R}$. For any $a, b \in \mathbb{R}$ and $\tau > 0$ let

$$[a, b]_\tau = [a, b] \cap \mathbb{R}_{a\tau} \,.$$

The functions $\mathrm{B}_{j,2}$, $\mathrm{B}_{j,3}$, and B_j with $\tau_j^{L_i} = \tau_j^{R_i} = \tau$, $j = i, i+1$ for all i are nonnegative on the discrete interval $[a, b]_\tau$. This permits us to reprove the main results for discrete polynomial splines of [9] for series of discrete generalized splines. Even more, one can obtain the results of generalized splines of [5] from the corresponding statements for discrete generalized splines as a limiting case when $\tau \to 0$.

In particular, if in (14) and (23) we have the coefficients $b_j^{(k)} > 0$, $k = 0, 1, 2$, $j = -3 + k, \ldots, N - 1$, then the spline S will be a positive, monotonically increasing and convex function on $[a, b]_\tau$.

Let f be a function defined on the discrete set $[a, b]_\tau$. We say that f has a zero at the point $x \in [a,b]_\tau$ provided

$$f(x) = 0 \quad \text{or} \quad f(x-\tau)\cdot f(x) < 0.$$

When f vanishes at a set of consecutive points of $[a,b]_\tau$, say f is 0 at $x, \ldots, x+(r-1)\tau$, but $f(x-\tau)\cdot f(x+r\tau) \neq 0$, then we call the set $X = \{x, x+\tau, \ldots, x+(r-1)\tau\}$ a *multiple zero* of f, and we define its multiplicity by

$$Z_X(f) = \begin{cases} r, & \text{if } f(x-\tau)\cdot f(x+r\tau) < 0 \text{ and } r \text{ is odd,} \\ r, & \text{if } f(x-\tau)\cdot f(x+r\tau) > 0 \text{ and } r \text{ is even,} \\ r+1, & \text{otherwise.} \end{cases}$$

This definition assures that f changes sign at a zero if and only if the zero is of odd multiplicity.

Let $Z_{[a,b]_\tau}(f)$ be the number of zeros of a function f on the discrete set $[a,b]_\tau$, counted according to their multiplicity. Let us denote $D_1^L S(x) = S[x-\tau, x]$.

Theorem 8 (Rolle's Theorem For Discrete Generalized Splines.) *For any $S \in S_4^{DG}$,*

$$Z_{[a,b]_\tau}(D_1^L S) \geq Z_{[a,b]_\tau}(S) - 1. \tag{24}$$

Proof: First, if S has a z-tuple zero on the set $X = \{x, \ldots, x + (r-1)\tau\}$, it follows that $D_1^L S$ has a $(z-1)$-tuple zero on the set $X' = \{x + \tau, \ldots, x + (r-1)\tau\}$. Now if X^1 and X^2 are two consecutive zero sets of S, then it is trivially true that $D_1^L S$ must have a sign change at some point between X^1 and X^2. Counting all of these zeros, we arrive at the assertion (24). This completes the proof. ♠

Lemma 9 *Let the function $D_{i,2}\Phi_i$ and $D_{i,2}\Psi_i$ be strictly monotone on the interval $[x_i, x_{i+1}]$ for all i. Then for every $S \in S_4^{DG}$ which is not identically zero on any interval $[x_i, x_{i+1}]_\tau$, $i = 0, \ldots, N-1$,*

$$Z_{[a,b]_\tau}(S) \le N + 2.$$

Proof: According to (19) and (23), the function $D_2 S$ has no more than one zero on $[x_i, x_{i+1}]$, because the functions $D_2\Phi_i$ and $D_2\Psi_i$ are strictly monotone and nonnegative on this interval. Hence $Z_{[a,b]_\tau}(D_2 S) \le N$. Then according to the Rolle's Theorem 8, we find $Z_{[a,b]_\tau}(S) \le N + 2$. This completes the proof. ♠

Denote by $\mathrm{supp}_\tau \mathrm{B}_i = \{x \in \mathbb{R}_{a,\tau} \,|\, \mathrm{B}_i(x) > 0\}$ the discrete support of the spline B_i, i.e. the discrete set $(x_i + \tau, x_{i+4} - \tau)_\tau$.

Theorem 10 *Assume that $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ are prescribed points on the discrete line $\mathbb{R}_{a,\tau}$. Then*

$$D = \det(\mathrm{B}_i(\zeta_j)) \ge 0, \quad i, j = -3, \ldots, N-1$$

and strict positivity holds if and only if

$$\zeta_i \in \operatorname{supp}_\tau \mathrm{B}_i, \quad i = -3, \dots, N-1. \tag{25}$$

The proof of this theorem is based on Lemma 9 and repeats that of Theorem 8.66 in [9, p.355]. The following statements follow immediately from Theorem 10.

Corollary 11 *The system of discrete* GB*-splines* $\{\mathrm{B}_j\}$, $j = -3, \dots, N-1$, *associated with knots on* $\mathbb{R}_{a,\tau}$ *is a weak Chebyshev system according to the definition given in [9, p. 36], i.e. for any* $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{N-1}$ *in* $\mathbb{R}_{a,\tau}$ *we have* $D \geq 0$ *and* $D > 0$ *if and only if condition (25) is satisfied. In the latter case the discrete generalized spline* $S(x) = \sum_{j=-3}^{N-1} b_j \mathrm{B}_j(x)$ *has no more than* $N + 2$ *zeros.*

Corollary 12 *If the conditions of Theorem 5 are satisfied, then the solution of the interpolation problem*

$$S(\zeta_i) = f_i, \quad i = -3, \dots, N-1, \quad f_i \in \mathbb{R} \tag{26}$$

exists and is unique.

Let $A = \{a_{ij}\}$, $i = 1, \dots, m$, $j = 1, \dots, n$, be a rectangular $m \times n$ matrix with $m \leq n$. The matrix A is said to be totally nonnegative (totally positive) (e.g., see [3]) if the minors of all order of the matrix are nonnegative (positive), i.e. for all $1 \le p \le m$ we have

$$\det(a_{i_k j_l}) \ge 0 \ (>0) \quad \text{for all} \qquad \begin{array}{l} 1 \le i_1 < \cdots < i_p \le m, \\ 1 \le j_1 < \cdots < j_p \le n. \end{array}$$

Corollary 13 *For arbitrary integers $-3 \le \nu_{-3} < \cdots < \nu_{p-4} \le N-1$ and $\zeta_{-3} < \zeta_{-2} < \cdots < \zeta_{p-4}$ in $\mathbb{R}_{a,\tau}$ we have*

$$D_p = \det\{\mathrm{B}_{\nu_i}(\zeta_j)\} \ge 0, \quad i,j = -3,\ldots,p-4$$

and strict positivity holds if and only if

$$\zeta_i \in \operatorname{supp}_\tau \mathrm{B}_{\nu_i}, \quad i = -3,\ldots,p-4$$

i.e. the matrix $\{\mathrm{B}_j(\zeta_i)\}$, $i,j = -3,\ldots,N-1$ is totally nonnegative.

The last statement is proved by induction based on Theorem 5 and the recurrence relations for the minors of the matrix $\{\mathrm{B}_j(\zeta_i)\}$. The proof does not differ from that of Theorem 8.67 described by [9, p.356].

Since the supports of discrete GB-splines are finite, the matrix of system (26) is banded and has seven nonzero diagonals in general. The matrix is tridiagonal if $\zeta_i = x_{i+2}$, $i = -3,\ldots,N-1$.

An important particular case of the problem, in which $S'(x_i) = f_i'$, $i = 0, N$, can be obtained by passing to the limit as $\zeta_{-3} \to \zeta_{-2}$, $\zeta_{N-1} \to \zeta_{N-2}$.

De Boor and Pinkus [2] proved that linear systems with totally nonnegative matrices can be solved by Gaussian elimination without choosing a pivot element. Thus, the system (26) can be solved effectively by the conventional Gauss method.

Denote by $S^-(\mathbf{v})$ the number of sign changes (variations) in the sequence of components of the vector $\mathbf{v} = (v_1, \cdots, v_n)$, with zeros being neglected. Karlin [3] showed that if a matrix A is totally nonnegative then it decreases the variation, i.e.

$$S^-(A\mathbf{v}) \le S^-(\mathbf{v}).$$

By virtue of Corollary 4, the totally nonnegative matrix $\{\mathrm{B}_j(\zeta_i)\}$, $i, j = -3, \ldots, N-1$, formed by discrete GB-splines decreases the variation.

For a bounded real function f, let $S^-(f)$ be the number of sign changes of the function f on the real axis $\mathbb{R}$, without taking into account the zeros

$$S^-(f) = \sup_n S^-[f(\zeta_1), \ldots, f(\zeta_n)], \quad \zeta_1 < \zeta_2 < \cdots < \zeta_n.$$

Theorem 14 *The discrete generalized spline $S(x) = \sum_{j=-3}^{N-1} b_j \mathrm{B}_j(x)$ is a variation diminishing function, i.e. the number of sign changes of S does* *not exceed that in the sequence of its coefficients:*

$$S^-\Big(\sum_{j=-3}^{N-1} b_j \mathrm{B}_j\Big) \le S^-(\mathbf{b}), \quad \mathbf{b} = (b_{-3}, \dots, b_{N-1}).$$

The proof of this statement does not differ from that of Theorem 8.68 for discrete polynomial B-splines in [9, p.356].

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References

[1] P. Costantini, B. I. Kvasov, and C. Manni. On discrete hyperbolic tension splines. *Advances in Computational Mathematics*, 11:331–354, 1999. C878, C881

[2] C. De Boor and A. Pinkus. Backward error analysis for totally positive linear systems. *Numer. Math.*, 27:485–490, 1977. C897

[3] S. Karlin. *Total Positivity, Volume 1.* Stanford University Press, Stanford, 1968. C896, C897

[4] P. E. Koch and T. Lyche. Exponential B-splines in tension. In C.K. Chui, L.L. Schumaker, and J.D. Ward, editors, *Approximation Theory VI: Proceedings of the Sixth International Symposium on Approximation Theory. Vol. II*, pages 361–364, 1989. Academic Press. C889

[5] B. I. Kvasov. Local bases for generalized cubic splines. *Russ. J. Numer. Anal. Math. Modelling*, 10:49–80, 1995. C881, C886, C892

[6] B. I. Kvasov. GB-splines and their properties. *Annals of Numerical Mathematics*, 3:139–149, 1996. C881

[7] B. I. Kvasov. Algorithms for shape preserving local approximation with automatic selection of tension parameters. *Computer Aided Geometric Design*, 17:17–37, 2000. C878

[8] T. Lyche. *Discrete polynomial spline approximation methods*. PhD thesis, University of Texas, Austin, 1975. For a summary, see *Spline Functions*, K. Böhmer, G. Meinardus, and W. Schimpp, editors, Karlsruhe 1975, Lectures Notes in Mathematics No. 501, pages 144–176, Springer-Verlag, Berlin, 1976. C881

[9] L. L. Schumaker. *Spline Functions: Basic Theory*. John Wiley & Sons, New York, 1981. C881, C892, C895, C895, C896, C898

[10] Yu. S. Zav'yalov, B. I. Kvasov, and V. L. Miroshnichenko. *Methods of Spline Functions*. Nauka, Moscow (in Russian), 1980. C881