Efficient solvers for the shallow water equations on a sphere

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Abstract

We present a finite element method using spherical splines to solve the shallow water equations on a sphere involving satellite data. We compare the proposed method with a meshless method using radial basis functions. The use of either radial basis functions or spherical splines leads to ill-conditioned systems of linear equations. To accelerate the solution process we use additive Schwarz and alternate triangular preconditioners. Some numerical experiments are presented to show the effectiveness of both preconditioners.

Keywords: Shallow water equations, radial basis function, spherical spline, preconditioning

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1 Introduction

An accurate weather prediction is very important for our society. It affects various industries such as agriculture, transportation and civil safety. Global weather prediction models are time consuming because high accuracy is required to give correct forecasts. Hence it is important to develop new efficient numerical methods. Before facing real life problems these methods should be tested on simplified models. One such model is the shallow water model [19]. The shallow water equations (SWEs) are a system of partial differential equations (PDEs) describing the water flow in ocean currents, coastal areas and river channels. The main property of this system is that the vertical scale is much smaller than the horizontal scale. This property allows one to average out the vertical components. In this article we use spherical splines and radial basis functions as approximate solutions of the SWEs on the unit sphere.

In the study of global atmospheric behaviour it is critical to solve PDEs on a sphere as this models the Earth's surface. When the given data (that is, initial conditions) involve scattered data, radial basis functions (RBFs) are especially suitable to approximate the solutions of the PDEs as they do not require any mesh generation. Recently a collocation method using RBFs was proposed for the spherical SWEs [5]. Another possible way to deal with scattered data is to use spherical splines [1, 3] (in the sense of Schumaker). Pham et al. [15] designed a Galerkin method using spherical splines to approximate the solutions of pseudodifferential equations on the unit sphere.

In this article the Galerkin formulation is used to discretise the spherical SWEs in space. To advance the resulting systems of ordinary differential equations (ODEs) in time a standard leap-frog scheme with implicit viscosity terms is used. In the Galerkin formulation, we propose two different finite dimensional spaces which are defined respectively by RBFs and spherical splines. This results in two different solution processes, one is a meshless method and the other is a finite element method. The resulting systems of linear equations are symmetric and positive definite, and are solved with the conjugate gradient method.

Whether RBFs or spherical splines are used to define the finite dimensional space, the resulting system is ill-conditioned. Hence preconditioners are needed to overcome this problem. Recently additive Schwarz preconditioners were used to solve pseudodifferential equations on the unit sphere with RBFs [12, 18] and with spherical splines [14, 16]. Another kind of preconditioner, the alternate triangular preconditioner, was proposed by Samarskii [17] to solve the Poisson equation with a finite difference method on the unit square. In this article we study the use of these preconditioners for the SWEs on the unit sphere. Some numerical experiments are presented to show the effectiveness of both preconditioners.

2 Spherical shallow water problem

The viscous SWEs on the rotating unit sphere $\mathbb{S}^2$ in Cartesian coordinates are [19]

$$\begin{aligned} \mathbf{V}_t + \mathbf{P}_\mathbf{x}\left[(\mathbf{V}\cdot\hat{\nabla})\mathbf{V} + f\mathbf{x}\times\mathbf{V} + g\hat{\nabla}\xi - \nu\nabla_s^2\mathbf{V}\right] = 0\,, \\ \xi_t + \mathbf{V}\cdot\hat{\nabla}\xi + \xi\hat{\nabla}\cdot\mathbf{V} - \nu\nabla_s^2\xi = 0\,, \end{aligned} \tag{1}$$

where all terms are defined in Table 1 and with initial conditions

$$\mathbf{V}(\,\cdot\,,0) = \mathbf{V}^0 \quad \text{and} \quad \xi(\,\cdot\,,0) = \xi^0\,. \tag{2}$$

The diffusive terms (those involving ν) are added to both equations of motion and the continuity equation to prevent spurious accumulation of energy and entropy at the model grid scale [7, 6, 13, 19]. The projection operator $\mathbf{P}_\mathbf{x}$ is used to limit the computation on the surface of the sphere. In the shallow water model the velocity vector is tangential to the surface of the sphere $\mathbb{S}^2$ so $\mathbf{P}_\mathbf{x}\mathbf{V} = \mathbf{V}$ [19]. Also, $\mathbf{P}_\mathbf{x}(\mathbf{x}\times\mathbf{V}) = \mathbf{x}\times\mathbf{V}$ and $\mathbf{P}_\mathbf{x}\hat{\nabla}\xi = \hat{\nabla}\xi$.

A weak formulation for (1) and (2) is to find the height $\xi(\,\cdot\,,t)\in H^1$ and velocity $\mathbf{V}(\,\cdot\,,t)\in\mathbf{H}^1$ for all $t\in[0,T]$ such that for all $w\in H^1$, $\mathbf{w}\in\mathbf{H}^1$ and $t\in[0,T]$

$$\begin{aligned} \langle\mathbf{V}_t,\mathbf{w}\rangle + \nu\langle\nabla_s(\mathbf{P}_\mathbf{x}\mathbf{V}),\nabla_s(\mathbf{P}_\mathbf{x}\mathbf{w})\rangle &= -\langle\mathbf{P}_\mathbf{x}(\mathbf{V}\cdot\hat{\nabla})\mathbf{V},\mathbf{w}\rangle \\ &\quad - \langle f\mathbf{x}\times\mathbf{V},\mathbf{w}\rangle - g\langle\hat{\nabla}\xi,\mathbf{w}\rangle\,, \\ \langle\xi_t,w\rangle + \nu\langle\nabla_s\xi,\nabla_s w\rangle &= -\langle\mathbf{V}\cdot\hat{\nabla}\xi,w\rangle - \langle\xi\hat{\nabla}\cdot\mathbf{V},w\rangle\,, \end{aligned} \tag{3}$$

with the initial conditions

$$\langle\mathbf{V}(\,\cdot\,,0),\mathbf{w}\rangle = \langle\mathbf{V}^0,\mathbf{w}\rangle \quad \text{and} \quad \langle\xi(\,\cdot\,,0),w\rangle = \langle\xi^0,w\rangle\,. \tag{4}$$

Here H^1 is the usual Sobolev space on the unit sphere $\mathbb{S}^2$ and $\langle\cdot,\cdot\rangle$ is the corresponding inner product. The bold font is used for spaces of vector-valued functions, whereas usual font is used for spaces of scalar-valued functions.

Table 1: Define the notation in terms of Cartesian coordinates $x = r\cos\theta\cos\varphi$, $y = r\cos\theta\sin\varphi$, $z = r\sin\theta$, and projection operator row vectors $\mathbf{p}_x = [1-x^2, -xy, -xz]$, $\mathbf{p}_y = [-xy, 1-y^2, -yz]$, $\mathbf{p}_z = [-xz, -yz, 1-z^2]$.

$\mathbf{x} := [x, y, z]^T$	position vector,
$\|\mathbf{x}\|_e := r = \sqrt{x^2+y^2+z^2}$	Euclidean norm of $\mathbf{x}$
$\mathbb{S}^2 := \{\mathbf{x} : \|\mathbf{x}\|_e = 1\}$	unit sphere centred at origin,
$\mathbf{V} := \mathbf{V}(\mathbf{x}, t) := (U, V, W)^T$	velocity in Cartesian coordinates,
$\xi := \xi(\mathbf{x}, t)$	height field,
$\Omega :=$ constant > 0	rotation rate of the Earth,
$f := f(z) := 2\Omega z$	Coriolis parameter,
$g :=$ constant > 0	gravitational constant,
$\nu :=$ constant $\geqslant 0$	viscosity of the fluid,
$\mathbf{P}_\mathbf{x} := [\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_z]^T$	projection operator matrix,
$\nabla := \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]^T$	Cartesian gradient operator,
$\hat{\nabla} := \mathbf{P}_\mathbf{x}\nabla$	Cartesian surface gradient operator,
$\nabla_s := \left[\frac{\partial}{\partial\theta}, \frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}\right]^T$	spherical surface gradient operator.

3 Galerkin equations

Let us denote by $\mathcal{V}$ a finite dimensional subspace of H^1 and by $\{B_1, \ldots, B_M\}$ a basis for $\mathcal{V}$. A Galerkin formulation for (3) and (4) is to find $\xi_h(\,\cdot\,, t) \in \mathcal{V}$ and $\mathbf{V}_h(\,\cdot\,, t) \in \boldsymbol{\mathcal{V}}$ for all $t \in [0, T]$ such that for all $w \in \mathcal{V}$, $\mathbf{w} \in \boldsymbol{\mathcal{V}}$ and $t \in [0, T]$

$$\begin{aligned}
\langle \mathbf{V}_{h,t}, \mathbf{w}\rangle + \nu\langle \nabla_s(\mathbf{P}_\mathbf{x}\mathbf{V}_h), \nabla_s(\mathbf{P}_\mathbf{x}\mathbf{w})\rangle &= -\langle \mathbf{P}_\mathbf{x}(\mathbf{V}_h\cdot\hat{\nabla})\mathbf{V}_h, \mathbf{w}\rangle \\
&\quad - \langle f\mathbf{x}\times\mathbf{V}_h, \mathbf{w}\rangle - g\langle\hat{\nabla}\xi_h, \mathbf{w}\rangle\,, \qquad (5)\\
\langle \xi_{h,t}, w\rangle + \nu\langle\nabla_s\xi_h, \nabla_s w\rangle &= -\langle \mathbf{V}_h\cdot\hat{\nabla}\xi_h, w\rangle - \langle \xi_h\hat{\nabla}\cdot\mathbf{V}_h, w\rangle\,,
\end{aligned}$$

with the initial conditions

$$\langle \mathbf{V}_h(\,\cdot\,, 0), \mathbf{w}\rangle = \langle \mathbf{V}^0, \mathbf{w}\rangle \quad\text{and}\quad \langle \xi_h(\,\cdot\,, 0), w\rangle = \langle \xi^0, w\rangle\,. \qquad (6)$$

Let us introduce the coefficient vectors $\mathbf{c}^{\mathbf{V}}(t) := [\mathbf{c}_1^{\mathbf{V}}(t), \ldots, \mathbf{c}_M^{\mathbf{V}}(t)]^T$ and $\mathbf{c}^{\xi}(t) := [c_1^{\xi}(t), \ldots, c_M^{\xi}(t)]^T$. The finite element approximation of the velocity and the height are, respectively,

$$\mathbf{V}_h(\mathbf{x}, t) = \sum_{i=1}^{M} \mathbf{c}_i^{\mathbf{V}}(t) B_i(\mathbf{x}) \quad \text{and} \quad \xi_h(\mathbf{x}, t) = \sum_{i=1}^{M} c_i^{\xi}(t) B_i(\mathbf{x}) \,.$$

By choosing $\mathbf{w}$ and w in (5) to be $\mathbf{B}_j = (B_j, B_j, B_j)^T$ and B_j, respectively, we obtain the following systems of ODEs:

$$\mathcal{M}\frac{d\mathbf{c}^{\mathbf{V}}}{dt} + \nu\mathcal{S}\mathbf{c}^{\mathbf{V}} = \mathcal{F}^{\mathbf{V}} \quad \text{and} \quad \mathbf{M}\frac{d\mathbf{c}^{\xi}}{dt} + \nu\mathbf{S}\mathbf{c}^{\xi} = \mathbf{F}^{\xi} \,. \tag{7}$$

Here $\mathbf{M} = (M_{ij})$ with $M_{ij} = \langle B_i, B_j \rangle$, $\mathbf{S} = (S_{ij})$ with $S_{ij} = \langle \nabla_s B_i, \nabla_s B_j \rangle$,

$$\mathcal{M} = \begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M} \end{bmatrix} \quad \text{and} \quad \mathcal{S} = \begin{bmatrix} \mathbf{S}^{U} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}^{W} \end{bmatrix} ,$$

with

$$\begin{aligned} S_{ij}^{U} &= \langle \nabla_s(\mathbf{p}_x \mathbf{B}_i), \nabla_s(\mathbf{p}_x \mathbf{B}_j) \rangle \,, \quad S_{ij}^{V} = \langle \nabla_s(\mathbf{p}_y \mathbf{B}_i), \nabla_s(\mathbf{p}_y \mathbf{B}_j) \rangle \,, \\ \text{and} \quad S_{ij}^{W} &= \langle \nabla_s(\mathbf{p}_z \mathbf{B}_i), \nabla_s(\mathbf{p}_z \mathbf{B}_j) \rangle \,. \end{aligned}$$

The right hand sides of (7) are $\mathcal{F}^{\mathbf{V}} = (\mathbf{F}^{U}, \mathbf{F}^{V}, \mathbf{F}^{W})^T$, with components

$$\begin{aligned} F_i^{U} &= -\langle \mathbf{p}_x(\mathbf{V}_h \cdot \hat{\nabla})\mathbf{V}_h, B_i \rangle - \langle f(yW_h - zV_h), B_i \rangle - g\langle \mathbf{p}_x \nabla \xi_h, B_i \rangle \,, \\ F_i^{V} &= -\langle \mathbf{p}_y(\mathbf{V}_h \cdot \hat{\nabla})\mathbf{V}_h, B_i \rangle - \langle f(zU_h - xW_h), B_i \rangle - g\langle \mathbf{p}_y \nabla \xi_h, B_i \rangle \,, \\ F_i^{W} &= -\langle \mathbf{p}_z(\mathbf{V}_h \cdot \hat{\nabla})\mathbf{V}_h, B_i \rangle - \langle f(xV_h - yU_h), B_i \rangle - g\langle \mathbf{p}_z \nabla \xi_h, B_i \rangle \,, \end{aligned}$$

and

$$F_i^{\xi} = -\langle \mathbf{V}_h \cdot \hat{\nabla} \xi_h, B_i \rangle - \langle \xi_h \hat{\nabla} \cdot \mathbf{V}_h, B_i \rangle \,.$$

Let $I_N := \{t_0, t_1, \dots, t_N\}$ be a partition of the interval $[0, T]$ where elements of partition are given by $t_n := nk$ with $n = 0, 1 \dots, N$, and $k := T/N$ is a time step. Let

$$\mathbf{V}_h^n(\mathbf{x}) = \mathbf{V}_h(\mathbf{x}, t_n) \quad \text{and} \quad \xi_h^n(\mathbf{x}) = \xi_h(\mathbf{x}, t_n),$$

and let us denote the corresponding coefficient vectors as $\mathbf{c}^{\mathbf{V},n}$ and $\mathbf{c}^{\xi,n}$, respectively. To discretise the systems of ODEs (7) in time, we use a standard leap-frog scheme with semi-implicit viscosity terms. Hence the time derivatives are approximated by central differences, the right hand sides are treated explicitly while the terms involving stiffness matrices $\mathcal{S}$ and $\mathbf{S}$ are treated semi-implicitly. This leads us to the following systems of linear equations:

$$\begin{aligned}
\mathcal{M}\left[\frac{\mathbf{c}^{\mathbf{V},n+1} - \mathbf{c}^{\mathbf{V},n-1}}{2k}\right] + \frac{\nu}{2}\mathcal{S}\mathbf{c}^{\mathbf{V},n+1} &= \mathcal{F}^{\mathbf{V},n} - \frac{\nu}{2}\mathcal{S}\mathbf{c}^{\mathbf{V},n-1}, \\
\mathbf{M}\left[\frac{\mathbf{c}^{\xi,n+1} - \mathbf{c}^{\xi,n-1}}{2k}\right] + \frac{\nu}{2}\mathbf{S}\mathbf{c}^{\xi,n+1} &= \mathbf{F}^{\xi,n} - \frac{\nu}{2}\mathbf{S}\mathbf{c}^{\xi,n-1},
\end{aligned}$$

which are rewritten as

$$\mathcal{A}\mathbf{c}^{\mathbf{V},n+1} = \mathcal{M}\mathbf{c}^{\mathbf{V},n-1} + 2k\mathcal{F}^{\mathbf{V},n} - k\nu\mathcal{S}\mathbf{c}^{\mathbf{V},n-1}, \tag{8}$$

$$\mathbf{A}\mathbf{c}^{\xi,n+1} = \mathbf{M}\mathbf{c}^{\xi,n-1} + 2k\mathbf{F}^{\xi,n} - k\nu\mathbf{S}\mathbf{c}^{\xi,n-1}, \tag{9}$$

where $\mathcal{A} = \mathcal{M} + k\nu\mathcal{S}$ and $\mathbf{A} = \mathbf{M} + k\nu\mathbf{S}$.

4 Finite dimensional subspaces

We now introduce finite dimensional spaces $\mathcal{V}$ defined by both spherical splines and RBFs. We denote by $\mathbf{X}$ a set of points on $\mathbb{S}^2$:

$$\mathbf{X} := \{\mathbf{x}_1, \dots, \mathbf{x}_M\} \subseteq \mathbb{S}^2. \tag{10}$$

4.1 Spherical splines

Following Schumaker [1] we introduce the space of spherical splines. Given a set of linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$, any vector $\mathbf{e} \in \mathbb{R}^3$ is uniquely represented as $\mathbf{e} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. The quantities b_1, b_2 and b_3, being linear homogeneous functions of $\mathbf{e}$, are called the trihedral coordinates of $\mathbf{e}$ with respect to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. The associated trihedron with the vertices $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ is the set

$$\tau := \{\mathbf{e} \in \mathbb{R}^3 : b_1(\mathbf{e}), b_2(\mathbf{e}), b_3(\mathbf{e}) \geqslant 0\}. \tag{11}$$

Given a non-negative integer d, the functions

$$B^{\tau,d}_{ijk}(\mathbf{e}) := \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k \quad \text{for } i+j+k=d\,,$$

are called the homogeneous Bernstein basis polynomials of degree d defined on τ. The functions $B^{\tau,d}_{ijk}$ are linearly independent and span the $\binom{d+2}{2}$ dimensional space $\mathcal{H}_d$ of homogeneous polynomials of degree d. A function $p_\tau(\mathbf{e}) = \sum_{i+j+k=d} c^\tau_{ijk} B^{\tau,d}_{ijk}(\mathbf{e})$, with $c^\tau_{ijk} \in \mathbb{R}$, is called a homogeneous Bernstein–Bézier (HBB) polynomial of degree d.

Let τ be a trihedron as in (11). Then the set $\tau \cap \mathbb{S}^2$ is a spherical triangle (henceforth we call it just triangle and refer to it as τ). The intersection of a spherical triangle τ and the plane passing through the origin and two vertices of τ will be called an edge of τ. Now we call a spherical Bernstein–Bézier (SBB) polynomial the restriction to the sphere $\mathbb{S}^2$ of an HBB polynomial. We write $\mathcal{P}_d$ for $\mathcal{H}_d$ restricted to the sphere $\mathbb{S}^2$.

Recalling (10), let

$$\Delta_h := \{\tau_i\}_{i=1}^{\mathcal{T}} \tag{12}$$

be a triangulation of $\mathbb{S}^2$ generated from the set $\mathbf{X}$, that is each τ_i is a spherical triangle with vertices as elements of $\mathbf{X}$ such that $\mathbb{S}^2 = \bigcup_i \bar{\tau}_i$ and that any two triangles intersect only at a common vertex or a common edge. Let $d \geqslant 1$ be an integer. Then we define

$$S_h := \{s \in C(\mathbb{S}^2) : s|_{\tau_i} \in \mathcal{P}_d, i = 1, \ldots, \mathcal{T}\}, \tag{13}$$

which is called the space of spherical splines of degree d.

The meshsize h of the triangulation Δ_h is defined as follows: for any spherical triangle $\tau \in \Delta_h$, we denote by $|\tau|$ the diameter of the smallest spherical cap containing τ. We define $|\Delta_h| := \max\{|\tau| : \tau \in \Delta_h\}$ and the meshsize $h := \tan(|\Delta_h|/2)$. We assume that the triangulation Δ_h is quasi-uniform and regular. Hence there exist positive constants $\beta > 1$ and $\gamma < 1$ such that

$$\gamma|\Delta| \leqslant |\tau| \leqslant \beta\rho_\tau\,, \quad \text{for all } \tau \in \Delta_h\,,$$

where ρ_τ is the diameter of the largest spherical cap inside τ.

4.2 Radial basis functions

Let us now introduce the space of RBFs $S^\phi_{\mathbf{X}}$. A real valued function in the form $\Phi(\mathbf{x}) = \phi(r)$, where $r = \|\mathbf{x}\|_e$, whose values depend only on the distance from the origin is called a radial basis function. Some commonly used RBFs are multiquadric $\phi(r) = \sqrt{1 + (\varepsilon r)^2}$, inverse quadratic $\phi(r) = [1 + (\varepsilon r)^2]^{-1}$ and Gaussian $\phi(r) = \exp[-(\varepsilon r)^2]$. A function in the form $\Phi_i(\mathbf{x}) = \phi(\|\mathbf{x} - \mathbf{x}_i\|_e)$ is called a radial basis function corresponding to the node $\mathbf{x}_i \in \mathbf{X}$ for $i = 1, \ldots, M$. Our finite dimensional space is then

$$S^\phi_{\mathbf{X}} := \operatorname{span}\{\Phi_1, \ldots, \Phi_M\}\,.$$

In our numerical simulations we use multiquadric radial basis functions with $\varepsilon = 3.25$, as those used by Flyer [5].

5 Preconditioners

The matrices $\mathbf{A}$ and $\mathcal{A}$ in (8)–(9) are symmetric and positive definite for both methods using spherical splines and RBFs. We use the conjugate gradient (CG) method to solve equations (8)–(9). If the condition numbers of these matrices are large, then the CG method converges slowly. To accelerate the convergence of the CG method we apply two different preconditioners, namely the additive Schwarz preconditioner and the alternate triangular preconditioner, which we introduce in the next two subsections.

5.1 Additive Schwarz preconditioner

To obtain the solution of (8)–(9) with the additive Schwarz method we solve the problems of smaller sizes independently. We employ this preconditioner only when $\mathcal{V} = S_h$. We denote $V = S_h$ and decompose it into a sum of subspaces so that

$$V = V_0 + \cdots + V_J\,. \tag{14}$$

We define the projection $P_j : V \to V_j$, for $j = 0, \dots, J$, by

$$a(P_j v, w) = a(v, w) \quad \text{for all } v \in V \text{ and for all } w \in V_j\,,$$

where the bilinear form a is $a(v,w) = \langle v, w\rangle + 2k\nu\langle \nabla_s v, \nabla_s w\rangle$. The additive Schwarz operator is then $P := P_0 + \cdots + P_J$. The solution of (8)–(9) is equivalent to the solution of the system $Pu_h = g$, where $g = \sum_{j=0}^{J} g_j$ and $g_j \in V_j$ is the solution of

$$a(g_j, w) = \langle F, w\rangle \quad \text{for all } w \in V_j\,.$$

To define V_j we first define a decomposition of $\mathbb{S}^2$. We recall the definitions (10), (12) and (13) for $\mathbf{X}$, Δ_h and S_h. We denote by $\mathbf{Y}$ some subset of $\mathbf{X}$. Let Δ_H be a triangulation built on the set $\mathbf{Y}$ with the property $H > h$ and let S_H be the space of spherical splines defined on Δ_H. To obtain the decomposition (14) we first define a coarse space $V_0 := \tilde{I}_h S_H$, where $\tilde{I}_h$ is a quasi-interpolant operator [11, p. 425]. For each $\tau_j^H \in \Delta_H$ where $j = 1, \dots, J$ we define the corresponding subdomain as

$$\Omega_j = \bigcup \{\tau \in \Delta_h : \bar{\tau} \cap \bar{\tau}_j^H \neq \varnothing\}\,.$$

Now we define the subspace V_j as a span of basis functions whose supports are subsets of Ω_j.

5.2 Alternate triangular preconditioner

In the alternate triangular preconditioning [9] we decompose the matrix $\mathbf{A}$ of the system of the linear equations in the form

$$\mathbf{A}\mathbf{c} = \mathbf{F} \tag{15}$$

into a sum of two matrices

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2\,, \quad \mathbf{A}_1^* = \mathbf{A}_2\,,$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are the lower and upper triangles of the matrix $\mathbf{A}$ and $\mathbf{A}_1^*$ is the conjugate transpose matrix of the matrix $\mathbf{A}_1$. Then we define the preconditioner

$$\mathbf{C}(\omega) = [(\mathbf{I} + \omega\mathbf{A}_1)(\mathbf{I} + \omega\mathbf{A}_2)]^{-1}\,, \quad \omega > 0\,, \tag{16}$$

where $\mathbf{I}$ is the identity operator. We employ the alternate triangular preconditioner for both cases when $\mathcal{V} = S_h$ and $\mathcal{V} = S_{\mathbf{X}}^{\phi}$. Let us denote by δ_1 and δ_2 the minimal and maximal eigenvalues of the matrix $\mathbf{A}$, respectively. It was proved [8] that the condition number of the matrix $\mathbf{C}(\omega)\mathbf{A}$ is bounded by $\eta(\omega) = 2\omega\delta_1/(1 + \omega\delta_1 + \omega^2\delta_1\delta_2/4)$. The parameter ω is then chosen to minimise the function $\eta(\omega)$. It was proved [9] that the minimiser is $\omega_* = 2/\sqrt{\delta_1\delta_2}$. Therefore, we choose $\mathbf{C}(\omega_*)$ as the optimal preconditioner. However, finding the eigenvalues δ_1 and δ_2 is a more complicated problem than solving the original problem (15). Konovalov [8] showed that if $\{\mathbf{c}^l\}$ are the iterates obtained by solving (15) with the steepest descend method and if $\hat{\omega}_l = \|\mathbf{c}^l\|_e/\|\mathbf{A}_2\mathbf{c}^l\|_e$, then $\kappa(\mathbf{C}(\hat{\omega}_l)\mathbf{A}) < \eta(\omega_*)$ for all l, where $\kappa(\mathbf{C}(\hat{\omega}_l)\mathbf{A})$ is the condition number of the matrix $\mathbf{C}(\hat{\omega}_l)\mathbf{A}$. Konovalov [10] also proved that the sequence $\{\hat{\omega}_l\}$ converges at a high rate. Numerical experiments show that the optimal value for $\kappa(\mathbf{C}(\hat{\omega}_l)\mathbf{A})$ is $\kappa(\mathbf{C}(\omega_{\text{opt}})\mathbf{A})$, where $\omega_{\text{opt}} = \lim_{l\to\infty}\hat{\omega}_l$. The obtained value ω_{opt} is then used to solve (15) with the preconditioned conjugate gradient method with the optimal preconditioner $\mathbf{C}(\omega_{\text{opt}})$. Table 2 gives a pseudocode for the proposed method to find ω_{opt}. In this pseudocode the iteration stops when there is convergence of $\{\hat{\omega}_l\}$, not of $\{\mathbf{c}^l\}$.

Table 2: A pseudocode of 12 steps for determining ω_{opt} .

(1)	$\hat{\omega} = 1, \hat{\omega}_{\text{old}} = 0, \mathbf{c} = \mathbf{F}$	(7)	$\mathbf{w} = (\mathbf{I} + \hat{\omega}\mathbf{A}_2)^{-1}\bar{\mathbf{w}}$
(2)	**while** $\lvert\hat{\omega} - \hat{\omega}_{\text{old}}\rvert > \text{tol}$	(8)	$\tau = \langle \mathbf{r}, \mathbf{w}\rangle_e / \langle \mathbf{A}\mathbf{w}, \mathbf{w}\rangle_e$
(3)	$\mathbf{r} = \mathbf{F} - \mathbf{A}\mathbf{c}$	(9)	$\mathbf{c} = \mathbf{c} + \tau\langle \mathbf{r}, \mathbf{w}\rangle_e \mathbf{w}$
(4)	$\hat{\omega}_{\text{old}} = \hat{\omega}$	(10)	$\text{iter} = \text{iter} + 1$
(5)	$\hat{\omega} = \lVert\mathbf{c}\rVert_e / \lVert\mathbf{A}_2\mathbf{c}\rVert_e$	(11)	**end while**
(6)	$\bar{\mathbf{w}} = (\mathbf{I} + \hat{\omega}\mathbf{A}_1)^{-1}\mathbf{r}$	(12)	$\omega_{\text{opt}} = \hat{\omega}$

6 Numerical experiments

In this section we compare the method using spherical splines and the method using RBFs for solving the spherical SWEs problem. We also compute the condition numbers of the system (8)–(9) for both methods and show the effectiveness of the additive Schwarz and the alternate triangular preconditioners. In our experiments we use data points from NASA's satellite MAGSAT to define the set of points $\mathbf{X}$. The initial conditions for the system (1)–(2) are proposed by Galewsky et al. [6].

In Table 3 we present the relative L^2 error of the height ξ for the method using RBFs and the method using spherical splines with piecewise linear, quadratic and cubic splines. The degree of freedom (DoF) for spherical splines is given by the number of domain points $\mathcal{D} := 2 + d^2(M-2)$, where d is the degree of splines and M is given by (10) [2]. On the problems with the same degrees of freedom, quadratic spherical splines give better approximation than RBFs. Usually for the same degrees of freedom we expect higher accuracy when using splines of higher degrees. However, in the case of homogeneous spherical splines, the spline spaces of even and odd degrees have only zero function in common. Therefore, in such spaces we cannot reproduce polynomials of degree $m \leqslant d$ unless $m \equiv d \mod 2$ [4]. This is the reason why we have better results for quadratic splines compared to cubic splines in Table 3.

In Tables 4–7 we show the condition numbers and the number of iterations for the CG method to converge for the system (8)–(9) and the preconditioned

Table 3: Relative L^2 error of the height ξ

DoF	101	204	414	836	1635	3250
RBFs	1.5E−2	3.6E−3	6.3E−4	1.8E−4	2.2E−5	1.6E−5
linear	3.6E−2	1.3E−2	8.4E−3	4.4E−3	2.2E−3	1.1E−3
DoF	94	194	398	810	1650	3338
quad.	4.3E−3	1.1E−3	2.3E−4	4.7E−5	1.6E−5	5.3E−6
DoF	92	209	434	893	1820	3710
cubic	4.2E−2	8.8E−3	1.6E−3	4.7E−4	8.7E−5	2.1E−5

Table 4: Condition numbers and number of iterations for multiquadric RBFs

DoF	101	204	414	836	1635	3250
κ	7.0E1	6.5E2	2.1E4	1.0E6	3.2E8	1.9E10
N	69	152	484	2060	15291	114891
κ_1	1.1E1	6.7E1	7.2E2	6.9E4	8.3E5	3.1E8
N_1	27	60	156	625	3834	35918

systems. We denote by κ the condition number of the unpreconditioned system, by κ_1 the condition number of the preconditioned system with the alternate triangular preconditioner and by κ_2 the condition number of the preconditioned system with the additive Schwarz preconditioners. We denote by N the number of iterations for the CG method to converge for the unpreconditioned systems (8)–(9), and by N_1 and N_2 the corresponding numbers of iterations for the systems with alternate triangular and additive Schwarz preconditioner, respectively. The matrices in (8)–(9) for the method using RBFs are much more ill-conditioned than those for the method using spherical splines. As expected, the condition numbers of preconditioned systems are smaller than those for unpreconditioned systems. The convergence rate of the preconditioned CG method is higher in all cases.

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Table 5: Condition numbers and number of iterations for piecewise linear splines

DoF	101	204	414	836	1635	3250
κ	1.9E1	4.0E1	7.3E1	2.8E2	6.0E2	1.1E3
N	64	92	128	184	254	353
κ_1	6.9E0	1.3E1	3.6E1	8.4E1	2.2E2	5.6E2
N_1	24	31	75	121	165	197
κ_2	1.1E1	2.9E1	4.1E1	4.3E1	4.7E1	5.1E1
N_2	51	67	95	113	133	134

Table 6: Condition numbers and number of iterations for piecewise quadratic splines

DoF	94	194	398	810	1635	3338
κ	1.1E1	2.4E1	4.7E1	1.8E2	2.0E2	7.6E2
N	56	78	106	158	206	296
κ_1	3.1E0	4.2E0	9.6E0	1.9E1	4.7E1	1.2E2
N_1	10	16	25	45	121	168
κ_2	9.2E0	1.9E1	3.4E1	4.0E1	4.4E1	4.8E1
N_2	42	54	93	110	125	132

Table 7: Condition numbers and number of iterations for piecewise cubic splines

DoF	92	209	434	893	1820	3710
κ	3.3E1	2.4E1	4.3E1	1.4E2	1.7E2	4.7E2
N	80	89	110	146	200	274
κ_1	5.2E0	1.5E1	3.4E1	4.7E1	9.5E1	2.0E2
N_1	19	54	98	107	106	171
κ_2	2.8E1	1.5E1	3.1E1	1.0E2	1.4E2	1.6E2
N_2	67	59	69	112	121	148

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