Fast iterative solvers for boundary value problems on a local spherical region

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(Received 29 October 2012; revised 5 August 2013)

Abstract

Boundary value problems on local spherical regions arise naturally in geophysics and oceanography when scientists model a physical quantity on large scales. Meshless methods using radial basis functions provide a simple way to construct numerical solutions with high accuracy. However, the linear systems arising from these methods are usually ill-conditioned, which poses a challenge for iterative solvers. We construct preconditioners based on an additive Schwarz method to accelerate the solution process for solving boundary value problems on local spherical regions.

Subject class: 65N35, 65N55

Keywords: boundary value problem, unit sphere, additive Schwarz method

Contents

1 Introduction

Let Ω be a simply connected local region with a smooth boundary $\partial\Omega$ on the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Let L be a differential operator, and let f and g be two given functions in certain Sobolev spaces. We assume that the boundary value problem

$$\begin{aligned} Lu &= f \text{ on } \Omega\,, \\ u &= g \text{ on } \partial\Omega\,, \end{aligned} \tag{1}$$

has a unique solution and that L is self-adjoint.

Such boundary value problems arise naturally in geophysics and oceanography when scientists model a physical quantity on large scales. In these situations, the curvature of the Earth cannot be ignored, and a boundary value problem has to be formulated on a local region of the unit sphere. For example, the study of planetary scale oceanographic flows in which oceanic eddies interact with topography such as ridges and land masses, or evolve in a closed

basin, leads to the study of point vortices on the surface of the sphere with boundaries [2, 6]. Such vortex motions are described as a Dirichlet problem on a subdomain of the sphere with the Laplace–Beltrami operator [1, 3].

Using a meshless method, we construct numerical solutions to (1) based on spherical radial basis functions (RBFs). To accelerate the solution process, we introduce a preconditioner based on the additive Schwarz method. In previous studies we considered pseudo-differential equations defined on the whole sphere [4, 7], in this article we focus on boundary value problems defined on local spherical regions.

2 Spherical RBFs

We assume that $\Phi : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{R}$ is a strictly positive definite kernel on $\mathbb{S}^n$, that is

- Φ is continuous,
- $\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^n$,
- For any set of distinct points $X = \{\mathbf{x}_1, \dots, \mathbf{x}_K\} \subset \mathbb{S}^n$, the matrix $[\Phi(\mathbf{x}_p, \mathbf{x}_q)]$ is strictly positive definite.

For a fixed point $\mathbf{x}_p \in \mathbb{S}^n$, the function $\Phi_p(\mathbf{x}) := \Phi(\mathbf{x}_p, \mathbf{x})$ is called a spherical RBF.

For mathematical analysis, sometimes it is convenient to expand the kernel Φ into a series of spherical harmonics [5]. The space of spherical harmonics of degree ℓ on $\mathbb{S}^n$, denoted by $\mathcal{H}_\ell$, has an orthonormal basis $\{Y_{\ell,k} : k = 1, \dots, N(n, \ell)\}$, where

$$N(n, 0) = 1 \quad \text{and} \quad N(n, \ell) = \frac{(2\ell + n - 1)\Gamma(\ell + n - 1)}{\Gamma(\ell + 1)\Gamma(n)} \text{ for } \ell \geqslant 1 .$$

The kernel Φ is expanded as

$$\Phi(\mathbf{x},\mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}), \quad \mathbf{x},\mathbf{y},\in \mathbb{S}^n,$$

where $\{\widehat{\phi}(\ell)\}_{\ell=0}^{\infty}$ is a sequence of positive real numbers such that $\sum_{\ell=0}^{\infty} N(n,\ell)\widehat{\phi}(\ell)$ is finite.

Every function $f \in L_2(\mathbb{S}^n)$ is also expanded in terms of spherical harmonics,

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(n,\ell)} \widehat{f}_{\ell,k} Y_{\ell,k}, \quad \widehat{f}_{\ell,k} = \int_{\mathbb{S}^n} f Y_{\ell,k}\, dS.$$

We define the inner product

$$\langle f, g\rangle_{\Phi} := \sum_{\ell=0}^{\infty} \frac{1}{\widehat{\phi}(\ell)} \sum_{k=0}^{N(n,\ell)} \widehat{f}_{\ell,k}\widehat{g}_{\ell,k},$$

and the associated norm $\|f\|_{\Phi} = \sqrt{\langle f, f\rangle_{\Phi}}$. Let

$$\mathcal{N}_{\Phi} := \{f \in L^2(\mathbb{S}^n) : \|f\|_{\Phi} < \infty\}.$$

It can be shown that $\mathcal{N}_{\Phi}$ is a reproducing kernel Hilbert space. For all $f \in \mathcal{N}_{\Phi}$,

$$\langle f, \Phi(\cdot,\mathbf{x})\rangle_{\Phi} = f(\mathbf{x}) \quad \text{for all } \mathbf{x} \in \mathbb{S}^n. \tag{2}$$

So Φ is the reproducing kernel of $\mathcal{N}_{\Phi}$.

3 A collocation method

Let $X^{(I)} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M\}$ be a set of scattered points in Ω and let $X^{(B)} = \{\mathbf{x}_{M+1}, \mathbf{x}_{M+2}, \ldots, \mathbf{x}_N\}$ be a set of scattered points on $\partial\Omega$. The uniformity

of the point set $X^{(I)}$ is measured by its mesh norm h_X and its separation radius q_X,

$$h_X := \sup_{y\in\Omega} \min_{x\in X} \cos^{-1}(x\cdot y) \quad \text{and} \quad q_X := \frac{1}{2} \min_{x,y\in\Omega\,,\,x\neq y} \cos^{-1}(x\cdot y)\,.$$

The angle $\cos^{-1}(x\cdot y)$ is the geodesic distance between two points x and y on the sphere.

Let

$$\varphi_j = \begin{cases} L_y\Phi(\cdot,x_j) & \text{for } j=1,\dots,M\,,\\ \Phi(\cdot,x_j) & \text{for } j=M+1,\dots,N\,. \end{cases}$$

Here L_x (or L_y) denotes the operator L acting on the first (or second) variable of the kernel $\Phi(x,y)$. We restrict ourselves to a class of rotational invariant operators such that $L_x\Phi(x,y) = L_y\Phi(x,y)$. This property holds for many operators frequently seen in practice, for example, the Laplace–Beltrami operator, the weakly-singular integral operator and the hypersingular integral operator [7]. Due to this assumption we simply write $L\Phi$ in place of $L_x\Phi$ or $L_y\Phi$.

Let

$$V = \operatorname{span}\{\varphi_1,\dots,\varphi_N\}\,. \tag{3}$$

The collocation method to solve (1) consists of finding a $u_X \in V$ that solves the collocation equations

$$Lu_X(x_k) = f(x_k) \quad \text{for } k=1,\dots,M\,, \tag{4a}$$
$$u_X(x_k) = g(x_k) \quad \text{for } k=M+1,\dots,N\,. \tag{4b}$$

In view of (1) and (3) we deduce from (4a)–(4b) that

$$u_X = \sum_{j=1}^{N} c_j\varphi_j\,,$$

where the coefficients c_j for $j = 1, \dots, N$ are determined from

$$\sum_{j=1}^{M} c_j LL\Phi(\mathbf{x}_k, \mathbf{x}_j) + \sum_{j=M+1}^{N} c_j L\Phi(\mathbf{x}_k, \mathbf{x}_j) = f(\mathbf{x}_k)\,, \quad k = 1, \dots, M\,,$$
$$\sum_{j=1}^{M} c_j L\Phi(\mathbf{x}_k, \mathbf{x}_j) + \sum_{j=M+1}^{N} c_j \Phi(\mathbf{x}_k, \mathbf{x}_j) = g(\mathbf{x}_k)\,, \quad k = M+1, \dots, N\,.$$

The above linear system in matrix form is

$$A\mathbf{c} = \mathbf{b}\,, \tag{5}$$

where

$$A = \begin{bmatrix} B_{LL} & B_L \\ B_L & B \end{bmatrix},$$

with

$$B_{LL} = [LL\Phi(\mathbf{x}_k, \mathbf{x}_j)]_{\mathbf{x}_k, \mathbf{x}_j \in X^{(I)}}\,,$$
$$B_L = [L\Phi(\mathbf{x}_k, \mathbf{x}_j)]_{\mathbf{x}_k \in X^{(I)}, \mathbf{x}_j \in X^{(B)}}\,,$$
$$B = [\Phi(\mathbf{x}_k, \mathbf{x}_j)]_{\mathbf{x}_k, \mathbf{x}_j \in X^{(B)}}\,,$$

and

$$\mathbf{b} = [f(\mathbf{x}_1) \dots f(\mathbf{x}_M)\,,\ g(\mathbf{x}_{M+1}) \dots g(\mathbf{x}_N)]^T\,.$$

The matrix A is symmetric positive definite, so an iterative method is used to solve for $\mathbf{c}$. The resulting linear system of equations is often ill-conditioned, especially when the minimum separation radius q_X is small. Before introducing fast iterative solvers for the linear system, we rewrite our collocation equations (4a)–(4b) as a variational problem.

Lemma 1. *Equations* (4a)–(4b) *are equivalent to*

$$\langle u_X, \varphi_j \rangle_\Phi = \langle u, \varphi_j \rangle_\Phi \quad \textit{for } j = 1, \dots, N\,. \tag{6}$$

Proof: For $j = 1, \ldots, M$, using (2),

$$\langle u_X, \varphi_j \rangle_\Phi = \langle u_X, L\Phi(\cdot, x_j) \rangle_\Phi = \langle Lu_X, \Phi(\cdot, x_j) \rangle_\Phi = Lu_X(x_j)$$

and

$$\left\langle L^{-1} f, \varphi_j \right\rangle_\Phi = \left\langle f, L^{-1} \varphi_j \right\rangle_\Phi = \left\langle f, L^{-1} L\Phi(\cdot, x_j) \right\rangle_\Phi = \langle f, \Phi(\cdot, x_j) \rangle_\Phi = f(x_j)\,,$$

so we rewrite (4a) as

$$\langle u_X, \varphi_j \rangle_\Phi = \left\langle L^{-1} f, \varphi_j \right\rangle_\Phi = \langle u, \varphi_j \rangle_\Phi\,. \tag{7}$$

Similarly, for $j = M+1, \ldots, N$, since $\langle u_X, \varphi_j \rangle_\Phi = \langle u_X, \Phi(\cdot, x_j) \rangle_\Phi = u_X(x_j)$ and $\langle g, \varphi_j \rangle_\Phi = \langle g, \Phi(\cdot, x_j) \rangle_\Phi = g(x_j)$ we rewrite (4b) as

$$\langle u_X, \varphi_j \rangle_\Phi = \langle g, \varphi_j \rangle_\Phi = \langle u, \varphi_j \rangle_\Phi\,. \tag{8}$$

♠

This lemma enables us to define the additive Schwarz method in the next section.

4 Additive Schwarz method

A framework for the additive Schwarz method applied to elliptic PDEs defined on the whole sphere without boundary conditions was discussed by Le Gia et al. [4]. In this section we propose a more general framework for boundary value problems on a subdomain of the unit sphere.

The additive Schwarz method provides a fast solution to equations (4a)-(4b) by solving, in parallel, problems of smaller size. Let the space V be decomposed as

$$V = V_0 + V_1 + \cdots + V_J + V_{J+1} + \cdots + V_K\,, \tag{9}$$

where we require that $V_k \subset \text{span}\{\varphi_1, \ldots, \varphi_M\}$ for $k = 0, \ldots, J$, and $V_k \subset \text{span}\{\varphi_{M+1}, \ldots, \varphi_N\}$ for $k = J+1, \ldots, K$.

For $k = 0, \ldots, K$ let $P_k : V \to V_k$ be defined by

$$\langle P_k w, \xi \rangle_\Phi = \langle w, \xi \rangle_\Phi \quad \text{for all } \xi \in V_k \text{ and for all } w \in V.$$

Let $P = P_0 + P_1 + \cdots + P_K$. The additive Schwarz method applied to the collocation equations involves solving

$$P u_X = h = \sum_{k=0}^{K} h_k, \tag{10}$$

where for $k = 0, \ldots, J$,

$$\langle h_k, \xi \rangle_\Phi = \left\langle L^{-1} f, \xi \right\rangle_\Phi \quad \text{for all } \xi \in V_k,$$

and for $k = J+1, \ldots, K$,

$$\langle h_k, \xi \rangle_\Phi = \langle g, \xi \rangle_\Phi \quad \text{for all } \xi \in V_k.$$

Lemma 2. *The approximate solution u_X is a solution of the variational equation (6) if and only if it is a solution of (10).*

Proof: Suppose u_X solves (6). Then using (7) for $k = 0, \ldots, J$,

$$\langle P_k u_X, \xi \rangle_\Phi = \langle u_X, \xi \rangle_\Phi = \left\langle L^{-1} f, \xi \right\rangle_\Phi = \langle h_k, \xi \rangle_\Phi \quad \text{for all } \xi \in V_k.$$

So, $P_k u_X = h_k$ for $k = 0, \ldots, J$. Similarly, for $k = J+1, \ldots, K$, and using (8),

$$\langle P_k u_X, \psi \rangle_\Phi = \langle u_X, \psi \rangle_\Phi = \langle g, \psi \rangle_\Phi = \langle h_k, \psi \rangle_\Phi \quad \text{for all } \psi \in V_k.$$

Hence

$$P u_X = \sum_{k=0}^{K} P_k u_X = \sum_{k=0}^{K} h_k = h,$$

that is, u_X satisfies (10). Conversely, suppose u_X solves (10). For $j = 1, \dots, N$,

$$\begin{aligned}\langle u_X, \varphi_j\rangle_\Phi &= \langle P^{-1}h, \varphi_j\rangle_\Phi = \langle h, P^{-1}\varphi_j\rangle_\Phi \\ &= \sum_{k=0}^{K} \langle h_k, P^{-1}\varphi_j\rangle_\Phi = \sum_{k=0}^{K} \langle h_k, P_k P^{-1}\varphi_j\rangle_\Phi \\ &= \sum_{k=0}^{J} \langle L^{-1}f, P_k P^{-1}\varphi_j\rangle_\Phi + \sum_{k=J+1}^{K} \langle g, P_k P^{-1}\varphi_j\rangle_\Phi \\ &= \sum_{k=0}^{K} \langle u, P_k P^{-1}\varphi_j\rangle_\Phi = \left\langle u, \sum_{k=0}^{K} P_k P^{-1}\varphi_j \right\rangle_\Phi = \langle u, \varphi_j\rangle_\Phi \ .\end{aligned}$$

In other words, u_X solves (6). ♠

To put the abstract framework of the additive Schwarz method into practice, we need to construct a concrete algorithm to decompose the space V appropriately. The decomposition is defined from decompositions of the sets of collocation points $X^{(I)}$ and $X^{(B)}$. The set $X^{(I)}$ is decomposed by the following algorithm.

1. Put Ω in a bounding box $E = [L_{\min}, L_{\max}] \times [l_{\min}, l_{\max}]$ in spherical or geographical coordinates.
2. Divide the box E into an $m \times n$ grid for some given positive integers m and n.
3. Enumerate the cells of the grid from 1 to $J := mn$.
4. Let $X_j := X^{(I)} \cap \mathrm{cell}(j)$ for $j = 1, \dots, J$.
5. From each X_j we choose one point which is closest to the centre of cell(j) to form the set X_0.

The set $X^{(B)} \subset \partial\Omega$ of boundary points is similarly divided into $K - J$ subsets.

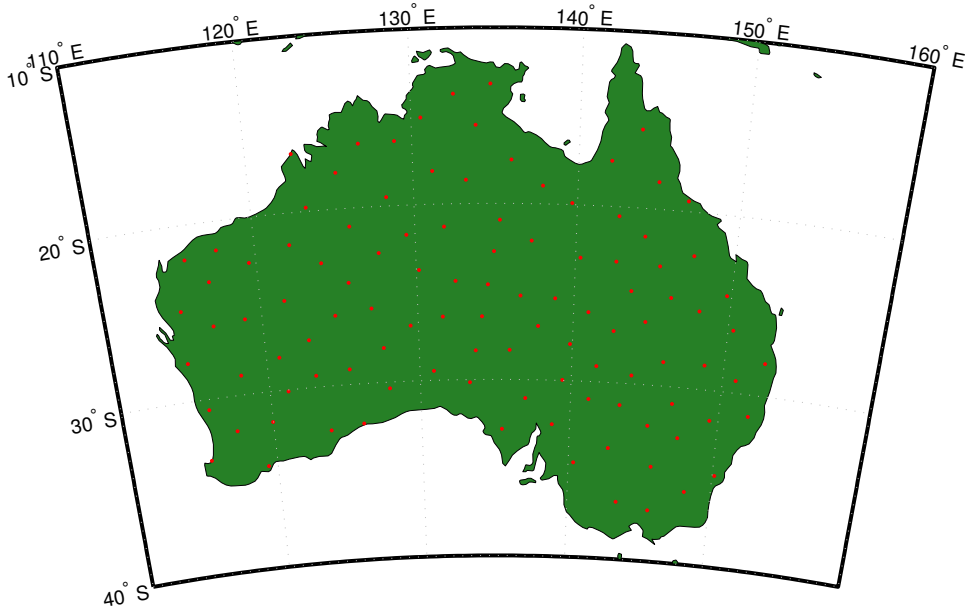

Figure 1: Bounding box of Australia and its decomposition into 15 cells (subdomains). The red dots form the set Ω and the grey dotted lines indicate the boundaries of the 15 cells.

To illustrate the algorithm, let Ω be the interior of Australia; see Figure 1. Firstly, the domain Ω is put into a bounding box $E = [110°E, 160°E] \times [10°S, 40°S]$, using geographical coordinates. Then E is sub-divided into $3 \times 5 = 15$ cells. The points inside the jth cell form the subset X_j, for $j = 1, \ldots, 15$. From each subset X_j, we choose one point which is closest to the centre of cell(j) to form the set X_0.

Given a partition of interior points $X^{(I)} = \bigcup_{j=0}^{J} X_j$, we define

$$V_j := \operatorname{span}\{\varphi_m = L\Phi(\cdot, x_m) : x_m \in X_j\} \quad \text{for } j = 0, \ldots, J.$$

Similarly, for the partition of the boundary points $X^{(B)} = \bigcup_{k=J+1}^{K} X_k$, we

define

$$V_k := \operatorname{span}\{\varphi_m = \Phi(\cdot, x_m) : x_m \in X_k\}, \quad k = J+1, \dots, K.$$

The additive Schwarz operator P is now a preconditioned solution operator. In terms of matrix equations,

$$P = MA = \begin{bmatrix} M_I & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} B_{LL} & B_L \\ B_L & B \end{bmatrix}.$$

In practice, we need to compute the action of M^{-1} on a residual $r \in V$. This consists of the solution of independent problems on each of the subspaces involved in the decomposition. The process is summarised in the following steps.

1. Correction of the global coarse set X_0; find $u_0 \in V_0$ satisfying

$$\langle u_0, \xi \rangle_\Phi = \langle r, \xi \rangle_\Phi \quad \text{for all } \xi \in V_0.$$

2. Corrections of the local interior sets X_j for $j = 1, \dots, J$; find $u_j \in V_j$ satisfying

$$\langle u_j, \xi \rangle_\Phi = \langle r, \xi \rangle_\Phi \quad \text{for all } \xi \in V_j.$$

3. Corrections of the local boundary sets X_k for $k = J+1, \dots, K$; find $u_k \in V_k$ satisfying

$$\langle u_k, \psi \rangle_\Phi = \langle r, \psi \rangle_\Phi \quad \text{for all } \psi \in V_k.$$

4. The residual in the conjugate gradient is preconditioned by

$$M^{-1} r := \sum_{j=0}^{J} u_j + \sum_{k=J+1}^{K} u_k.$$

5 Numerical experiments

Let us consider the boundary value problem

$$\begin{aligned} -\Delta^* u &= f \quad \text{in } \Omega\,, \\ u &= g \quad \text{in } \partial\Omega\,. \end{aligned} \tag{11}$$

Here Δ^* is the Laplace–Beltrami operator defined on the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. We choose f and g so that the exact solution is

$$u(\theta,\phi) = \sin\theta\cos\phi + [2\sin(2\theta) - \sin(4\theta)]\cos(3\phi)\,, \quad \theta \in [0,\pi]\,, \phi \in [0,2\pi)\,.$$

The RBF used in the experiments is $\Phi(\mathbf{x},\mathbf{y}) = \rho\left(\sqrt{2-2\mathbf{x}\cdot\mathbf{y}}\right)$ where $\rho(r) = (35r^2+18r+3)(1-r)_+^6$. The point sets are taken from MAGSAT satellite data restricted to Ω, where Ω is some local region of the Earth. In the first experiment we solve an academic problem on $\Omega = \Omega_1 = [110°\mathrm{E}, 160°\mathrm{E}] \times [10°\mathrm{S}, 40°\mathrm{S}]$ in geographical coordinates. In the second experiment we solve a more practical problem with $\Omega = \Omega_2$ being the interior of Australia (as in Figure 1).

We solve the matrix equation (5) using the conjugate gradient method with a relative tolerance of 10^{-7}, that is the stopping criterion is

$$\frac{\|A\mathbf{c}^{(m)} - \mathbf{b}\|_{\ell_2}}{\|\mathbf{b}\|_{\ell_2}} \leqslant 10^{-7}\,.$$

The conjugate gradient method is considered non-convergent when the number of iterations exceeds $20N$, where $N \times N$ is the dimension of the matrix A.

As is seen from the numerical results presented in Table 1, 2 and 3, the Schwarz preconditioner significantly improves the CPU time (measured in seconds) and reduces the number of iterations required for the conjugate gradient (CG) method. For the first experiment involving Ω_1, the unpreconditioned CG method applied to the problem is not convergent. For both examples, since the number of collocation points on the boundary $X^{(B)}$ is rather small, we did

Table 1: Numerical results for the boundary value problem (11) defined on Ω_1 using the preconditioned CG. The unpreconditioned CG does not converge in this example.

N	J	$\lambda_{\min}(P)$	$\lambda_{\max}(P)$	$\kappa(P)$	CPU	iter
10914	40	4.4E−3	8.80	2.0E+3	115	221
10914	50	4.4E−3	9.88	2.2E+3	115	235
10914	60	4.4E−3	10.96	2.5E+3	112	246
10914	70	5.2E−3	11.56	2.2E+3	95	236
10914	80	5.2E−3	12.42	2.4E+3	97	239
10914	90	5.0E−3	13.29	2.7E+3	107	241
12425	30	3.5E−3	7.70	2.2E+3	147	233
12425	40	3.9E−3	8.81	2.3E+3	160	238
12425	50	3.8E−3	9.91	2.6E+3	164	243
12425	60	3.7E−3	10.99	2.9E+3	166	262
12425	70	4.7E−3	11.60	2.5E+3	143	241
12425	80	4.6E−3	12.48	2.7E+3	147	254
12425	90	4.6E−3	13.33	2.9E+3	143	251
13739	40	3.6E−3	8.82	2.5E+3	170	233
13739	50	3.5E−3	9.94	2.8E+3	155	243
13739	60	3.4E−3	11.00	3.2E+3	156	270
13739	70	4.5E−3	11.63	2.6E+3	127	238
13739	80	4.4E−3	12.51	2.8E+3	126	252
13739	90	4.3E−3	13.36	3.1E+3	115	262

not decompose the space $\operatorname{span}\{\varphi_{M+1}, \ldots, \varphi_N\}$ and hence $K = J + 1$ in both cases. The condition number $\kappa(P)$ of the additive Schwarz operator P seems to be fairly independent of the number of subdomains J.

Table 2: Numerical results for (11) defined on Ω_2 using preconditioned CG.

N	J	$\lambda_{\min}(P)$	$\lambda_{\max}(P)$	$\kappa(P)$	CPU	iter
1493	10	1.2E−4	6.64	5.4E+4	11	497
1493	12	1.3E−4	6.78	5.3E+4	11	515
1493	14	1.3E−4	7.63	6.0E+4	10	527
1493	15	1.3E−4	7.42	5.5E+4	10	541
1493	16	1.3E−4	7.54	5.7E+4	10	537
1493	20	1.4E−4	8.19	5.9E+4	10	561
2026	10	9.1E−5	6.65	7.3E+4	23	560
2026	12	9.3E−5	6.81	7.3E+4	23	593
2026	14	9.5E−5	7.69	8.1E+4	20	617
2026	15	9.8E−5	7.44	7.6E+4	21	625
2026	16	9.7E−5	7.53	7.7E+4	21	621
2026	20	1.0E−4	8.18	8.0E+4	19	620
4054	10	3.1E−5	6.76	2.2E+5	228	1181
4054	12	3.2E−5	6.91	2.2E+5	235	1265
4054	14	3.3E−5	7.81	2.4E+5	185	1295
4054	15	3.3E−5	7.55	2.3E+5	189	1259
4054	16	3.3E−5	7.66	2.3E+5	194	1316
4054	20	3.4E−5	8.32	2.5E+5	175	1388

Table 3: Numerical results for (11) defined on Ω_2 using unpreconditioned CG.

N	$\lambda_{\min}(A)$	$\lambda_{\max}(A)$	$\kappa(A)$	CPU	iter
1493	0.39E−5	0.29E+7	0.73E+12	26	14846
2026	0.71E−5	0.40E+7	0.56E+12	48	14763
4054	0.42E−6	0.78E+7	0.18E+14	1017	71685

6 Conclusion

We suggested a collocation method with RBFs for a boundary value problem in a local region on a sphere. The method is designed in such a way that the resulting matrix is symmetric and positive definite. (The method is sometimes called the symmetric collocation method in the RBF literature.) However, as is well known, the matrix is ill-conditioned. We provided a remedy by using additive Schwarz preconditioners. The use of a symmetric collocation method requires that the preconditioner be designed properly so that the resulting system is still equivalent to the original problem. Lemma 2, which justifies this equivalence, can be extended to any subdomain of a general Riemannian manifold equipped with a reproducing Hilbert space structure.

We carried out numerical experiments on a practical domain (namely, the interior of the Australian continent) to support our theory. The numerical results showed that the RBF collocation method with a preconditioned conjugate gradient method is very competitive for solving boundary value problems on local spherical regions.

Acknowledgements The support of the Australian Research Council is gratefully acknowledged by both authors.

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