A vector spherical harmonic spectral code for linearised magnetohydrodynamics

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Abstract

Linearised rotating magnetohydrodynamic stability code for the steady axisymmetric basic states of an electrically conducting fluid sphere is described. The code generates compact hybrid angular spectral forms of the magnetic induction, heat and Boussinesq Navier-Stokes equations, using toroidal and poloidal representations of the perturbation vector fields, and vector or scalar spherical harmonic expansions of all fields. The momentum equation may include inertial, Coriolis, buoyancy, viscous and magnetic Lorentz forces. Three subroutines evaluate the spectral interactions of products. There are only six radial functions, which

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are discretised using uniform second-order finite differences. The resulting large scale complex non-hermitian generalised eigen- and critical-value problems are solved using inverse and Newton-Raphson iteration methods, respectively. Test results are presented for several models.

Contents

1 Introduction

The linearised magnetohydrodynamic stability of an electrically-conducting fluid sphere is considered. Such studies complement

non-linear dynamically-consistent dynamo calculations. The non-linear equations for an electrically-conducting Boussinesq fluid in a frame rotating with uniform angular velocity $\boldsymbol{\Omega}$ governing the velocity $\mathbf{v}$, the pressure p, the magnetic induction $\mathbf{B}$ and the temperature Θ are

$$\rho(\partial_t\mathbf{v} + \boldsymbol{\omega}\times\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v}) = -\nabla P + \mathbf{J}\times\mathbf{B} - \alpha_\Theta\Theta\mathbf{g} + \rho\nu\nabla^2\mathbf{v} \quad (1)$$

$$\partial_t\mathbf{B} = \eta\nabla^2\mathbf{B} + \nabla\times(\mathbf{v}\times\mathbf{B})\,, \quad (2)$$

$$\partial_t\Theta + \mathbf{v}\cdot\nabla\Theta = \kappa\nabla^2\Theta + Q/\rho c_p\,, \quad (3)$$

$$\nabla\cdot\mathbf{v} = 0\,, \quad \nabla\cdot\mathbf{B} = 0\,, \quad (4)$$

where $P = p + \frac{1}{2}\rho\mathbf{v}^2$ is the modified pressure, $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is the vorticity, $\mathbf{J} = \nabla\times\mathbf{B}/\mu_0$ is the electrical current, ν, η and κ are the viscous, magnetic and thermal diffusivities, α_Θ is the thermal expansivity and Q is the rate of heat production per unit volume and c_p is the specific heat at constant pressure. The gravitational field may be asymmetric. The diffusivities and specific heat are constants.

The rotating MHD equations (1)–(4), linearised about the steady axisymmetric basic state $(\mathbf{v}_0, \mathbf{B}_0, \Theta_0)$, with $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$, etc, are

$$\rho(\partial_t\mathbf{v}' + 2\boldsymbol{\Omega}\times\mathbf{v}') = -\nabla P' + \mathbf{J}_0\times\mathbf{B}' +$$
$$+ \mathbf{J}'\times\mathbf{B}_0 - \rho\alpha_\Theta\Theta'\mathbf{g} + \rho\nu\nabla^2\mathbf{v}'\,, \quad (5)$$

$$\partial_t\mathbf{B}' = \eta\nabla^2\mathbf{B}' + \nabla\times(\mathbf{v}_0\times\mathbf{B}') + \nabla\times(\mathbf{v}'\times\mathbf{B}_0)\,, \quad (6)$$

$$\partial_t\Theta' + \mathbf{v}_0\cdot\nabla\Theta' + \mathbf{v}'\cdot\nabla\Theta_0 = \kappa\nabla^2\Theta' + Q'/\rho c_p\,, \quad (7)$$

$$\nabla\cdot\mathbf{v}' = 0\,, \quad \nabla\cdot\mathbf{B}' = 0\,. \quad (8)$$

The interaction terms, which arise from the non-linear advection, $\boldsymbol{\omega}_0\times\mathbf{v}'$ and $\boldsymbol{\omega}'\times\mathbf{v}_0$, are not required for the test models considered in §4 and have been omitted. They may be treated similarly to the Lorentz terms. The equations must be supplemented by boundary conditions. Special cases include: thermal convection with $\mathbf{v}_0 = \mathbf{0}$,

Θ_0 the conduction temperature and $\mathbf{B}_0$ omitted; kinematic dynamos with $\mathbf{v}_0$ given, $\mathbf{B}_0 = \mathbf{0}$ and Θ_0 omitted; and magnetoconvection with $\mathbf{v}_0 = \mathbf{0}$, $\mathbf{B}_0$ and $\Theta_0(r)$ given. In stability problems equations (5)–(8), with $Q' = 0$, and the associated boundary conditions are homogeneous and have solutions $\mathbf{B}', \mathbf{v}', \Theta' \propto e^{\gamma t}$. The problem is then a generalised eigen- or critical-value problem (§4.1). Primes on perturbation fields are suppressed below.

The linearised equations are discretised in angle using vector spherical harmonic expansions (§2.2) of all vector fields and spherical harmonic expansions (§2.1) of all scalar fields, for both the basic and perturbation states. Toroidal-poloidal representations (§2.1) are also used, but only for the perturbation solenoidal vector fields, not the basic state fields. Thus there are five independent perturbation scalar fields: the temperature, and the toroidal and poloidal potentials of the magnetic field and the velocity. The resulting hybrid angular spectral equations (§2.3) are more compact and less error prone to code, than if the toroidal-poloidal spectral interactions of products are fully expanded. Moreover, although there are ten products, only three subroutines are required to evaluate their angular dependence (§3). The subroutines calculate the three angular coupling integrals, which occur, in terms of $3j$-, $6j$- and $9j$-symbols. Further, only six distinct combinations of perturbation radial functions and their derivatives occur, which greatly simplifies radial discretisation (§3). The code uses second-order finite-differences on a uniform radial grid. The resulting large-scale complex non-hermitian generalised eigen- and critical-value problems are solved using inverse and Newton-Raphson iteration methods, respectively. The code is tested against several models: rotating thermal convection (§4.1), kinematic roll-dynamos (§4.3) and magneto-convection (§4.4). Concluding remarks are given in §5.

2   Hybrid vector spherical harmonic method

2.1   Toroidal-poloidal fields and spherical harmonics

The solenoidal conditions (8) are satisfied by the toroidal-poloidal representations

$$\mathbf{B} = \mathbf{T}\{T\} + \mathbf{S}\{S\}, \quad \mathbf{v} = \mathbf{T}\{t\} + \mathbf{S}\{s\}, \tag{9}$$

where $\mathbf{T}\{T\} := \nabla \times \{T\mathbf{r}\}$ and $\mathbf{S}\{S\} := \nabla \times \mathbf{T}\{S\}$. From the identity $\nabla \times \mathbf{S}\{S\} = \mathbf{T}\{-\nabla^2 S\}$, it follows for the current and vorticity that

$$\mu_0 \mathbf{J} = \mathbf{T}\{-\nabla^2 S\} + \mathbf{S}\{T\}, \quad \boldsymbol{\omega} = \mathbf{T}\{-\nabla^2 s\} + \mathbf{S}\{t\}.$$

The perturbation scalar fields are expanded in terms of (scalar) spherical harmonics. The spherical harmonic of degree n and order m in colatitude θ and east-longitude ϕ is defined for $m \geq 0$ by

$$Y_n^m(\theta, \phi) := (-)^m \sqrt{\frac{(2n+1)(n-m)!}{(n+m)!}} P_{n,m}(\cos \theta) e^{im\phi},$$

where

$$P_{n,m}(z) = (-)^n \frac{(1-z^2)^{m/2}}{2^n n!} \frac{d^{m+n}(1-z^2)^n}{dz^{m+n}}$$

is the associated Legendre function of degree n and order m. For $m < 0$ the spherical harmonic of degree n and order m is defined by $Y_n^m := (-)^m (Y_n^{-m})^*$, where the star denotes complex conjugation. The spherical harmonics are orthonormal with respect to the inner-product, $(f, g) := \frac{1}{4\pi} \oint f g^* \, d\Omega$. A lowercase Greek subscript denotes

the 2-index of a spherical harmonic. For example, Y_α denotes $Y_{n_\alpha}^{m_\alpha}$ and the orthonormalisation condition is $(Y_\alpha, Y_\beta) = \delta_\alpha^\beta$. Thus the poloidal and toroidal potentials, pressure and temperature have the spherical harmonic expansions

$$f = \sum_\alpha f_\alpha Y_\alpha, \quad \text{for } f \in \{s, t, p, S, T, \Theta\}. \tag{10}$$

Summations are over $n_\alpha \in \mathbb{N}$, and $m_\alpha = -n_\alpha, \ldots, n_\alpha$.

2.2 Vector spherical harmonics

The vector spherical harmonics used herein are defined by James [5]

$$\mathbf{Y}_{n,n_1}^m := (-)^{n-m}\sqrt{2n+1} \sum_{m_1,\mu} \begin{pmatrix} n & n_1 & 1 \\ m & -m_1 & -\mu \end{pmatrix} Y_{n_1}^{m_1} \mathbf{e}_\mu,$$

where $\mathbf{e}_0 := \mathbf{1}_z$ and $\mathbf{e}_{\pm 1} := \mp(\mathbf{1}_x \pm i\mathbf{1}_y)/\sqrt{2}$. In particular, $\mathbf{Y}_{n,0}^\mu = \delta_n^1 \mathbf{e}_\mu$ and $\mathbf{Y}_{0,1}^0 = -\mathbf{1}_r$. A lowercase Greek subscript on a vector field or its coefficient or argument denotes the 3-index of a vector spherical harmonic. Thus $\mathbf{Y}_\gamma$ denotes $\mathbf{Y}_{n_\gamma,n_{\gamma 1}}^{m_\gamma}$, otherwise $\gamma\,(\gamma_1)$ denotes n_γ $(n_{\gamma 1})$. If a 3-index occurs on a scalar field or its coefficient the n_1 index is dropped. Vector spherical harmonics are orthonormal with respect to the inner-product $(\mathbf{F}, \mathbf{G}) := \frac{1}{4\pi}\oint \mathbf{F} \cdot \mathbf{G}^*\, d\Omega$ of complex vector functions $\mathbf{F}$ and $\mathbf{G}$ of θ and ϕ. Hence $(\mathbf{Y}_\alpha, \mathbf{Y}_\beta) = \delta_{\alpha,\beta}$. The basic-state and perturbation velocity, vorticity, magnetic induction and electric current fields are expanded in vector spherical harmonics,

$$\mathbf{F}_0 = \sum_\alpha F_\alpha^0 \mathbf{Y}_\alpha, \quad \mathbf{F} = \sum_\alpha F_\alpha \mathbf{Y}_\alpha, \quad \mathbf{F} = \mathbf{v}, \boldsymbol{\omega}, \mathbf{B}, \mathbf{J}. \tag{11}$$

The summations are over $n_\alpha \in \mathbb{N}$, $n_{1\alpha} = n_\alpha, n_\alpha \pm 1$ and $m_\alpha = -n_\alpha, \ldots, n_\alpha$.

The Y-expansions of t and s are related to the $\mathbf{Y}$-expansion of $\mathbf{v}$ by $v_\gamma = f_v(\gamma)\partial_\gamma s_\gamma$, if $\gamma_1 = \gamma \pm 1$, and $v_\gamma = f_v(\gamma)t_\gamma$, if $\gamma_1 = \gamma$, where

$$f_v(\gamma) := \begin{cases} (\gamma+1)\sqrt{\gamma/(2\gamma+1)}, & \text{if } \gamma_1 = \gamma - 1; \\ -i\sqrt{\gamma(\gamma+1)}, & \text{if } \gamma_1 = \gamma; \\ \gamma\sqrt{(\gamma+1)/(2\gamma+1)}, & \text{if } \gamma_1 = \gamma + 1; \end{cases} \tag{12}$$

and $\partial_n^{n_1} := \partial_r + (n+1)/r$, if $n_1 = n - 1$ and $\partial_n^{n_1} := \partial_r - n/r$, if $n_1 = n + 1$. In 3-index notation, $\partial_\gamma := \partial_{n_\gamma}^{n_1\gamma}$ and $\partial^\gamma := \partial_{n_1\gamma}^{n_\gamma}$. Similarly, the Y-expansions of T and S are related to the $\mathbf{Y}$-expansion of $\mathbf{B}$ by $B_\gamma = f_B(\gamma)\partial_\gamma S_\gamma$, if $\gamma_1 = \gamma \pm 1$, and $B_\gamma = f_B(\gamma)T_\gamma$, if $\gamma_1 = \gamma$, where $f_B(\gamma) = f_v(\gamma)$ for all γ. The $\mathbf{Y}$-components of the electric current are given by $\mu_0 J_\gamma = f_J(\gamma)\partial_\gamma T_\gamma$, if $\gamma_1 = \gamma \pm 1$, and $\mu_0 J_\gamma = f_J(\gamma)D_\gamma S_\gamma$, if $\gamma_1 = \gamma$, where

$$f_J(\gamma) := \begin{cases} (\gamma+1)\sqrt{\gamma/(2\gamma+1)}, & \text{if } \gamma_1 = \gamma - 1; \\ i\sqrt{\gamma(\gamma+1)}, & \text{if } \gamma_1 = \gamma; \\ \gamma\sqrt{(\gamma+1)/(2\gamma+1)}, & \text{if } \gamma_1 = \gamma + 1. \end{cases} \tag{13}$$

The $\mathbf{Y}$-components of the vorticity are related to the toroidal-poloidal velocity potentials by similar equations. The $\mathbf{Y}$-expansions of the temperature gradient $\mathbf{q}$ are related to the Y-expansion of Θ by $q_\gamma = f_q(\gamma)\partial_\gamma\Theta_\gamma$, if $\gamma_1 = \gamma \pm 1$, and $q_\gamma = 0$, if $\gamma_1 = \gamma$, where f_q is the temperature gradient field factor,

$$f_q(\gamma) := \begin{cases} \sqrt{\dfrac{\gamma}{2\gamma+1}}, & \text{if } \gamma_1 = \gamma - 1; \\ 0, & \text{if } \gamma_1 = \gamma; \\ -\sqrt{\dfrac{\gamma+1}{2\gamma+1}}, & \text{if } \gamma_1 = \gamma + 1. \end{cases} \tag{14}$$

2.3   The hybrid linearised spectral equations

The hybrid linearised spectral momentum equations [3] for t_γ and s_γ are

$$\rho(\partial_t - \nu D_\gamma)t_\gamma = \sum_{\substack{\alpha,\beta \\ (n_{1\gamma}=n_\gamma)}} (J^0_\alpha B_\beta v_\gamma) - (B^0_\beta J_\alpha v_\gamma)$$

$$- \rho\alpha_\Theta(g^0_\alpha \Theta_\beta v_\gamma) - \rho 2(\Omega_\alpha v_\beta v_\gamma), \quad (15)$$

$$\rho(\partial_t - \nu D_\gamma)D_\gamma s_\gamma = \sum_{\substack{\alpha,\beta \\ n_{1\gamma}(\neq n_\gamma)}} (J^0_\alpha B_\beta v_\gamma) - (B^0_\beta J_\alpha v_\gamma)$$

$$- \rho\alpha_\Theta(g_\alpha \Theta_\beta v_\gamma) - \rho 2(\Omega_\alpha v_\beta v_\gamma), \quad (16)$$

where $r^2 D_\gamma := r^2 \partial_{rr} + 2r\partial_r - n_\gamma(n_\gamma + 1)$.

The magnetic vector potential equation, $\partial_t \mathbf{A} = \eta \nabla^2 \mathbf{A} + \mathbf{v} \times \mathbf{B} + \nabla V$, where $\mathbf{B} = \nabla \times \mathbf{A}$, is used instead of the magnetic induction equation, since $\mathbf{F} \times \mathbf{G}$ is easier to handle than $\nabla \times (\mathbf{F} \times \mathbf{G})$. The magnetic vector potential spectral equations [3] for S_γ and T_γ are

$$(\partial_t - \eta D_\gamma)S_\gamma = \sum_{\substack{\alpha,\beta \\ (n_{1\gamma}=n_\gamma)}} (v^0_\alpha B_\beta A_\gamma) - (B^0_\alpha v_\beta A_\gamma), \quad (17)$$

$$(\partial_\tau - \eta D_\gamma)T_\gamma = \sum_{\substack{\alpha,\beta \\ n_{1\gamma}(\neq n_\gamma)}} (v^0_\alpha B_\beta A_\gamma) - (B^0_\alpha v_\beta A_\gamma). \quad (18)$$

The hybrid spectral linearised heat equation [3] for Θ_γ is

$$(\partial_t - \kappa D_\gamma)\Theta_\gamma = \sum_{\alpha,\beta} -(v^0_\alpha q_\beta \Theta_\gamma) - (q^0_\alpha v_\beta \Theta_\gamma). \quad (19)$$

The interaction terms are

$$
(J^0_\alpha B_\beta v_\gamma) = e_v(\gamma) f_B(\beta) (\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) \times
$$
$$
\times \begin{cases}
\partial^\gamma (J^0_\alpha \partial_\beta S_\beta), & \gamma_1 = \gamma \pm 1, \ \beta_1 = \beta \pm 1; \\
\partial^\gamma (J^0_\alpha T_\beta), & \gamma_1 = \gamma \pm 1, \ \beta_1 = \beta; \\
J^0_\alpha \partial_\beta S_\beta, & \gamma_1 = \gamma, \ \beta_1 = \beta \pm 1; \\
J^0_\alpha T_\beta, & \gamma_1 = \gamma, \ \beta_1 = \beta;
\end{cases}
$$

$$
(B^0_\alpha J_\beta v_\gamma) = e_v(\gamma) f_J(\beta) (\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma) \times
$$
$$
\times \begin{cases}
\partial^\gamma (B^0_\alpha \partial_\beta T_\beta), & \gamma_1 = \gamma \pm 1, \ \beta_1 = \beta \pm 1; \\
\partial^\gamma (B^0_\alpha D_\beta S_\beta), & \gamma_1 = \gamma \pm 1, \ \beta_1 = \beta; \\
B^0_\alpha \partial_\beta T_\beta, & \gamma_1 = \gamma, \ \beta_1 = \beta \pm 1; \\
B^0_\alpha D_\beta S_\beta, & \gamma_1 = \gamma, \ \beta_1 = \beta;
\end{cases}
$$

$$
(g_\alpha \Theta_\beta v_\gamma) = \rho \alpha_\Theta e_v(\gamma) (\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma) \begin{cases}
\partial^\gamma (g_\alpha \Theta_\beta), & \gamma_1 = \gamma \pm 1; \\
g_\alpha \Theta_\beta, & \gamma_1 = \gamma;
\end{cases}
$$

$$
(v^0_\alpha q_\beta \Theta_\gamma) = f_q(\beta) (\mathbf{Y}_\alpha \cdot \mathbf{Y}_\beta, Y_\gamma) \begin{cases}
v^0_\alpha \partial_\beta \Theta_\beta, & \beta_1 = \beta \pm 1; \\
0, & \beta_1 = \beta.
\end{cases}
$$

The terms $(\Omega^0_\alpha v_\beta v_\gamma)$, $(v^0_\alpha B_\beta A_\gamma)$ and $(B^0_\alpha v_\beta A_\gamma)$ have the same form as $(J^0_\alpha B_\beta v_\gamma)$, and are obtained by making the obvious substitutions. The $(q^0_\alpha v_\beta \Theta_\gamma)$ has the same form as $(v^0_\alpha q_\beta \Theta_\gamma)$. Each term is a product of an equation factor, a perturbation field factor, a coupling integral and a radial expression. The velocity, magnetic and temperature equation factors are

$$
e_v(\gamma) := \begin{cases}
1/\sqrt{\gamma(2\gamma + 1)}, & \text{if } \gamma_1 = \gamma - 1; \\
i/\sqrt{\gamma(\gamma + 1)}, & \text{if } \gamma_1 = \gamma; \\
1/\sqrt{(\gamma + 1)(2\gamma + 1)}, & \text{if } \gamma_1 = \gamma + 1;
\end{cases}
$$

$e_B(\gamma) = (-)^{\gamma - \gamma_1} e_v(\gamma)$ and $e_\Theta = 1$.

The coupling integrals of these three harmonics, $(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)$, $(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)$ and $(\mathbf{Y}_\alpha \cdot \mathbf{Y}_\beta, Y_\gamma)$, have been evaluated in closed form [5] in terms of $3j$-, $6j$- and $9j$-symbols, which are readily evaluated. The series (10) and (11), and the equations (15)–(19), are truncated at $n = N_n$.

3   Code structure and radial discretisation

There are three general routines for cross-products $\mathbf{F}_0 \times \mathbf{F}$, dot products $\mathbf{F}_0 \cdot \mathbf{F}$ and scalar multiplications $f\mathbf{F}_0$. The routine for cross-products calculates the angular part, that is, the coupling integral $(\mathbf{Y}_\alpha \times \mathbf{Y}_\beta, \mathbf{Y}_\gamma)$, and calls two common routines to evaluate the radial part and the position of an entry in the relevant matrix. This one routine evaluates all seven cross products, $\mathbf{v}_0 \times \mathbf{B}$, $\mathbf{v} \times \mathbf{B}_0$, $\mathbf{1}_z \times \mathbf{v}$, $\mathbf{J} \times \mathbf{B}_0$, $\mathbf{J}_0 \times \mathbf{B}$. The dot product and scalar multiplication routines calculate the coupling integrals, $(\mathbf{Y}_\alpha \cdot \mathbf{Y}_\beta, Y_\gamma)$ and $(\mathbf{Y}_\alpha Y_\beta, \mathbf{Y}_\gamma)$, but call the same radial and position routines. They evaluate $\mathbf{v}_0 \cdot \nabla\Theta$, $\mathbf{v} \cdot \nabla\Theta_0$ and $\mathbf{g}\Theta$. There is also one routine for the scalar and vector Laplacians, $\nabla^2\Theta$, $\nabla^2\mathbf{A}$, $\nabla^2\mathbf{v}$.

Two symmetries are exploited to reduce the numerical problem size. The first is the axisymmetry of the basic state, which decouples the spherical harmonic orders m of the perturbation fields. Eigen- and critical solutions are of the form, $f(r, \theta, \phi) = \widetilde{f}(r, \theta)e^{im\phi}$, for $f \in \{s, t, S, T, \Theta\}$. The order m enters as a parameter, which is fixed apriori, eliminating one independent variable, ϕ. Since the fields are real, only $m \geq 0$ is considered. The second is symmetry under reflection in the equator, that is, parity: even/odd (e/o) for scalars and dipole/quadrupole (d/q) for vectors. A scalar f is even/odd, if $f(r, \pi - \theta, \phi) = \pm f(r, \theta, \phi)$; a vector field $\mathbf{F}$ is dipole/quadrupole, if

$F_r(r, \pi - \theta, \phi) = \mp F_r(r, \theta, \phi)$, $F_\theta(r, \pi - \theta, \phi) = \pm F_\theta(r, \theta, \phi)$, $F_\phi(r, \pi - \theta, \phi) = \mp F_\phi(r, \theta, \phi)$. Since $Y_n^m(\pi - \theta, \phi) = (-)^{n-m} Y_n^m(\theta, \phi)$, the field $\mathbf{B}$ ($\mathbf{v}$) is dipole/quadrupole, if $S_n^m = 0$ ($s_n^m = 0$) for $n - m$ even/odd and $T_n^m = 0$ ($t_n^m = 0$) for $n - m$ odd/even. Θ is even/odd, if $\Theta_n^m = 0$ for $n - m$ odd/even. The problem decouples into two groups, $(\mathbf{B}_q, \mathbf{v}_q, \Theta_e)$ and $(\mathbf{B}_d, \mathbf{v}_q, \Theta_e)$, if $\mathbf{v}_d = \mathbf{0}$, $\Theta_o = 0$ and $\mathbf{g}_d = \mathbf{0}$.

Only six types of radial expressions occur,

$$f^0 \partial_\beta f, \ f^0 D_\beta f, \ \partial^\gamma(f^0 f), \ \partial^\gamma(f^0 \partial_\beta f), \ \partial^\gamma(f^0 D_\beta f), \ D_\gamma D_\gamma f,$$

apart from the trivial $f^0 f$. The compact form of these expressions greatly simplifies radial discretisation. Second-order finite-differences on a uniform grid, $r_j = j/N_r$, $j = 1, \ldots, N_r$, are used. The centred schemes,

$$f_0^{(1)} = \frac{-f_{-1} + f_1}{2h} - \frac{1}{6} f^{(3)}(\eta) h^2, \tag{20}$$

$$f_0^{(2)} = \frac{f_{-1} - 2f_0 + f_1}{h^2} - \frac{1}{12} f^{(4)}(\eta) h^2, \tag{21}$$

$$f_0^{(3)} = \frac{-f_{-2} + 2f_{-1} - 2f_1 + f_2}{2h^3} - \frac{1}{4} f^{(5)}(\eta) h^2, \tag{22}$$

$$f_0^{(4)} = \frac{f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2}{h^4} - \frac{1}{6} f^{(6)}(\eta) h^2, \tag{23}$$

where the subscripts indicate relative position, are used at the interior points r_j, where $j = 1, \ldots, N_r - 1$ for (20), (21) and $j = 1, \ldots, N_r - 2$ for (22), (23). The schemes must be adapted near $r = 0$. If a function f is analytic at the origin, then f_n^m is of the form $f_n^m = r^n \sum_{k=0}^\infty a_k r^{2k}$ near $r = 0$, where the a_k are independent of r, and hence $f_n^m(-r) = (-)^n f_n^m(r)$. This identity is used to fold the schemes from negative grid points back to positive points. Note that the poloidal-toroidal potentials of an analytic vector field are analytic.

The centred schemes are supplemented by one-sided right boundary schemes near $r = 1$,

$$f_0^{(1)} = \frac{f_{-2} - 4f_{-1} + 3f_0}{2h} + \frac{1}{3}f^{(3)}(\eta)h^2 \,, \tag{24}$$

$$f_0^{(2)} = \frac{-f_{-3} + 4f_{-2} - 5f_{-1} + 2f_0}{h^2} + \frac{11}{12}f^{(4)}(\eta)h^2 \,, \tag{25}$$

$$f_0^{(3)} = \frac{3f_{-4} - 14f_{-3} + 24f_{-2} - 18f_{-1} + 5f_0}{2h^3} + \frac{7}{4}f^{(5)}(\eta)h^2 \,, \tag{26}$$

$$f_0^{(3)} = \frac{f_{-3} - 6f_{-2} + 12f_{-1} - 10f_0 + 3f_1}{2h^3} + \frac{1}{4}f^{(5)}(\eta)h^2 \,, \tag{27}$$

if the boundary conditions fix f, and

$$f_0^{(2)} = \frac{-f_{-2} + 8f_{-1} - 7f_0 + 6hf_0^{(1)}}{2h^2} + \frac{1}{6}f^{(4)}(\eta)h^2 \,, \tag{28}$$

$$f_0^{(3)} = \frac{f_{-3} - 6f_{-2} + 15f_{-1} - 10f_0 + 6hf_0^{(1)}}{h^3} + \frac{11}{20}f^{(5)}(\eta)h^2 \,, \tag{29}$$

$$f_0^{(3)} = \frac{-f_{-2} + 9f_0 - 8f_1 + 6hf_1^{(1)}}{3h^3} - \frac{3}{20}f^{(5)}(\eta)h^2 \,, \tag{30}$$

$$f_0^{(4)} = \frac{-3f_{-3} + 32f_{-2} - 108f_{-1} + 192f_0 - 113f_1 + 60hf_1^{(1)}}{12h^4}$$
$$- \frac{1}{21}f^{(7)}(\eta)h^3 \,, \tag{31}$$

if the boundary conditions fix $f^{(1)}$. Schemes (24)–(26), (28) and (29) are applied at r_{N_r}; schemes (27), (30) and (31) are applied at r_{N_r-1}. The code produces three large, sparse, banded, complex matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. The bandwidth of the matrices for the most general problem is minimised, if the spherical harmonic coefficients, S_n^m, T_n^m, s_n^m, t_n^m, Θ_n^m, at each gridpoint are blocked together. If the problem uses the difference schemes (22) and (23), then the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are block pentadiagonal, apart from a possible third subdiagonal boundary block, if schemes (25), (27), (29) and (31) are used, and

fourth subdiagonal boundary block, if (26) is used. The matrices are $N \times N$, where

$$N = \sum_{f \in \{S,T,s,t,\Theta\}} (N_r - b_f)(N_n - m + 1)p_f\,,$$

with $b_f = 1$, if f is fixed at $r = 1$ by the boundary condition and $b_f = 0$ otherwise, and with $p_f = 0.5$ if f is even or odd and $p_f = 1$ otherwise.

4 Models and results

4.1 Methods for eigen- and critical-value problems

The code produces the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ for two kinds of problem. The first is the generalised eigenvalue problem,

$$(\mathbf{A} + R\mathbf{C})\mathbf{x} = \gamma\mathbf{B}\mathbf{x}\,, \tag{32}$$

for the growth rate γ of largest real part and the associated eigenvector $\mathbf{x}$, where R is a dimensionless parameter. The second is the critical value problem,

$$(\mathbf{A} + R\mathbf{C})\mathbf{x} = i\omega\mathbf{B}\mathbf{x}\,, \tag{33}$$

for the critical value of the real parameter R and the critical frequency $\omega = \Im\gamma$, that is, the values where the real part of the growth rate γ vanishes, $\Re\gamma = 0$. Problem (32) is solved using generalised inverse iteration,

$$\mathbf{y}_n = (\mathbf{A} + R\mathbf{C} - \gamma_0\mathbf{B})^{-1}\mathbf{B}\mathbf{x}_{n-1}\,, \quad \gamma_n = \gamma_0 + \hat{y}_n\,, \quad \mathbf{x}_n = \hat{y}_n^{-1}\mathbf{y}_n\,,$$

where $\hat{y}_n$, which may be complex, is the element of $\mathbf{y}_n$ of largest magnitude, and $(\gamma_0, \mathbf{x}_0)$ is an estimate of $(\gamma, \mathbf{x})$. Problem (33), which is non-linear in the unknowns, $(R, \omega, \mathbf{x})$, as well as under-determined since it is homogeneous in $\mathbf{x}$, is supplemented by an linear normalising equation in $\mathbf{x}$ and solved by Newton-Raphson iteration [7].

4.2   Rotating thermal convection

The basic state for rotating thermal convection is $\mathbf{B}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{0}$ and the steady conduction temperature Θ_0. In the present model, the volume heat production is constant, Θ_0 vanishes on the boundary $r = a$, and gravity $\mathbf{g} = -g_a\mathbf{r}/a$. In terms of the boundary temperature gradient $-\beta$, the heat production is $Q_0 = 3\kappa\rho c_p\beta/a$ and the temperature is $\Theta_0 = \beta(a^2 - r^2)/2a$. The perturbation magnetic field $\mathbf{B}$ simply decays, since $\mathbf{v}_0 = \mathbf{0}$, and so is omitted. Further, there is no surplus heat production, $Q = Q_0$.

The remaining problem is non-dimensionalised using the radial length scale a, the viscous diffusion time scale a^2/ν, the viscous velocity scale ν/a and the temperature scale $a\beta$. The dimensionless equations for the perturbation velocity and temperature, $\mathbf{v}$ and Θ are

$$E\partial_t\mathbf{v} + \mathbf{1}_z \times \mathbf{v} = -\nabla P + R\,\Theta\mathbf{r} + E\nabla^2\mathbf{v}\,,$$
$$\partial_t\Theta - \mathbf{v}\cdot\mathbf{r} = \mathrm{Pr}^{-1}\nabla^2\Theta\,,$$

where $E := \nu/2\Omega a^2$, $R := \alpha g_a a^2\beta/2\Omega\nu$ and $\mathrm{Pr} := \nu/\kappa$ are the Ekman, a modified Rayleigh and Prandtl numbers. On the boundary the velocity satisfies the no-slip condition, $\mathbf{v} = \mathbf{0}$, and the temperature vanishes, $\Theta = 0$. Thus

$$s_n^m = \partial_r s_n^m = t_n^m = \Theta_n^m = 0\,, \quad \text{at } r = 1\,.$$

TABLE 1: Results for rotating thermal convection model.

E	m	N_n	N_r	$\Im\gamma$	R	R [6]
10^{-2}	1	10	100	0.10434	180.74	180.74
10^{-2}	2	10	100	0.097868	157.83	157.88
10^{-2}	3	10	100	0.089180	186.20	186.25
5.7735×10^{-3}	1	10	200	0.065721	159.00	159.10
5.7735×10^{-3}	2	10	200	0.066732	129.02	129.06
5.7735×10^{-3}	3	10	200	0.070462	137.26	137.30
3.1623×10^{-3}	1	10	200	0.025556	143.94	143.98
3.1623×10^{-3}	2	10	200	0.029171	111.04	111.11
3.1623×10^{-3}	3	10	200	0.040854	111.96	112.02

Critical values of the modified Rayleigh number for the test model [6] with $\mathrm{Pr} = 1$ are shown in Table 1 for the quadrupole modes. The associated critical frequencies are also given. In [6], which treats only rotating thermal convection in a sphere, the toroidal-poloidal momentum equations and the heat equation are discretised using a Galerkin method in radius and angle. The radial basis functions are linear combinations of modified spherical Bessel functions and the angle basis functions are spherical harmonics. For a given radial truncation level the method of [6] is more accurate than the radial finite-difference/angle Galerkin method used herein, but it is also substantially more difficult to apply and is not as general.

4.3   Kinematic dynamos with axisymmetric flows

The basic state for kinematic dynamos is $\mathbf{B}_0 = \mathbf{0}$, $\Theta_0 = 0$ with a prescribed flow $\mathbf{v}_0$, which is axisymmetric. The magnetic field $\mathbf{B}$ is due to the electric currents generated by the motion $\mathbf{v}_0$ of a spher-

ical conducting fluid V of radius a and uniform magnetic diffusivity η. The exterior is electrically insulating and the dynamo is self-exciting, that is, there are no sources of magnetic field at infinity. The perturbation velocity, temperature, Navier-Stokes equation and heat equation are omitted. The equations are non-dimensionalised using the radial length scale a, a typical flow speed v_0 and the magnetic diffusion time scale, a^2/η. The magnetic field $\mathbf{B}$ is governed by

$$\partial_t \mathbf{B} = \nabla^2 \mathbf{B} + R\,\nabla \times (\mathbf{v}_0 \times \mathbf{B})\,, \quad \text{in } V\,;$$
$$\nabla \cdot \mathbf{B} = 0\,, \quad \text{in } E^3\,; \tag{34}$$
$$\nabla \times \mathbf{B} = \mathbf{0}\,, \quad \text{in } \widehat{V}\,; \tag{35}$$
$$[\mathbf{B}]_{\partial V} = \mathbf{0}\,, \quad \text{on } \partial V\,; \quad \mathbf{B} \to \mathbf{0}\,, \quad \text{as } r \to \infty\,; \tag{36}$$

where $R := a v_0/\eta$ is the magnetic Reynolds number.

The equations in the insulating exterior $\widehat{V}$ reduce to $T = \nabla^2 S = 0$ and the continuity condition on ∂V becomes $[T] = 0 = [S] = [\partial_r S]$ at $r = 1$. Substitution of the expansions (10) gives $T_n^m(r,t) = 0$ and $S_n^m(r,t) = S_n^m(1,t)r^{-n-1}$ in $r > 1$. Eliminating $S_n^m(1,t)$, using the continuity conditions, yields the local poloidal and toroidal boundary conditions,

$$\partial_r S_n^m + (n+1)S_n^m = T_n^m = 0\,, \quad \text{at } r = 1\,.$$

The magnetic (Ohmic) free-decay case $R_m = 0$ has analytic solution

$$T_{nk}^m = j_n(j_{n,k}r)e^{-j_{n,k}^2 t}\,, \quad S_{nk}^m = j_n(j_{n-1,k}r)e^{-j_{n-1,k}^2 t}\,,$$

for $r < 1$, $n, k \in \mathbb{N}^+$, where j_n is the nth order spherical Bessel function of the first kind and $j_{n,k}$ is the kth zero of j_n. The slowest decaying poloidal modes are the $n = 1$, $k = 1$ ($m = -1, 0, 1$) modes with (dimensional) decay time $t_P = a^2/j_{01}^2\eta$, $j_{01} = \pi$. The slowest

TABLE 2: Dynamo growth rates for flow $s_2 t_1$, with $m = 1$, $N_n = 12$ and $N_r = 100$.

ϵ	R	$\Re\gamma$	$\Im\gamma$	$\Re\gamma$ [1]	$\Im\gamma$ [1]
0.1	120	0.9955	-31.325	0.996	-31.31
0.13	94.9385	0	-18.2153		
0.13	100	0.5552	-18.7935		

decaying toroidal modes are the $n = 1$, $k = 1$ ($m = -1, 0, 1$) modes with decay time $t_T = a^2/j_{11}^2\eta$, $j_{11} = 4.493409$. (For the Earth $t_P = 19,500 \, \text{yr}$ and $t_T = 9,500 \, \text{yr}$ excluding a primordial magnetic field.)

The kinematic dynamo test model is a double-roll $s_2 t_1$ flow [1],

$$\mathbf{v}_0 = \mathbf{t}_1^0 + \epsilon \mathbf{s}_2^0, \quad t_1^0 = \sin(\pi r)/\sqrt{3}, \quad s_2^0 = r\sin(\pi r)/\sqrt{5},$$

where ϵ is a disposable parameter. The non-zero basic state vector spherical harmonic coefficients are

$$v_{1,1}^0 = -i\sqrt{2/3}\sin \pi r,$$
$$v_{2,1}^0 = \epsilon 3\sqrt{2}/5(\pi r \cos \pi r + 4\sin \pi r),$$
$$v_{2,3}^0 = \epsilon 2\sqrt{3}/5(\pi r \cos \pi r - \sin \pi r).$$

Selected results are shown in Table 2 for $m = 1$ quadrupole solutions. In [1] the magnetic induction equation is discretised using toroidal and poloidal fields, a spherical harmonic Galerkin method in angle and second-order finite-differences in radius with Richardson extrapolation, but a lower order radial boundary scheme. The critical magnetic Reynolds number 94.9385 for $\epsilon = 0.13$ compares with the value 95 given in [1].

4.4 Rotating magnetoconvection

The basic state for magnetoconvection is $\mathbf{v}_0 = \mathbf{0}$ with the steady conduction temperature Θ_0 and a prescribed magnetic field $\mathbf{B}_0$, which is axisymmetric. The thermal convection of a rotating spherical conducting fluid V of radius a, which is driven by the temperature gradient, is modified by the imposed magnetic field $\mathbf{B}_0$. The perturbation magnetic field is due to the electric currents in V. The diffusivities, ν, κ and η, are constant. The exterior is electrically insulating and the magnetic field perturbations are self-exciting. In the present model, the volume heat production is constant, Θ_0 vanishes on the boundary $r = a$ and gravity $\mathbf{g} = -g_a \mathbf{r}/a$. The temperature is $\Theta_0 = \beta(a^2 - r^2)/2a$. There is no surplus heat production, $Q = 0$.

The variables are non-dimensionalised using the radial length scale a, the magnetic diffusion time scale a^2/η, the magnetic velocity scale η/a, the temperature scale $a\beta$ and the magnetic scale B_0, where B_0 is a typical value of $\mathbf{B}_0$. The dimensionless equations for the perturbation velocity, magnetic field and temperature, $\mathbf{v}$, $\mathbf{B}$, Θ, are

$$E_\eta \partial_t \mathbf{v} + \mathbf{1}_z \times \mathbf{v} = -\nabla P + \Lambda \mathbf{J}_0 \times \mathbf{B} + R\Theta \mathbf{r} + E\nabla^2 \mathbf{v},$$
$$\partial_t \mathbf{B} = \nabla^2 \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}_0),$$
$$\partial_t \Theta - \mathbf{v} \cdot \mathbf{r} = q\nabla^2 \Theta,$$

where $E_\eta := \eta/2\Omega a^2$, $\Lambda := B_0^2/2\Omega\eta\mu_0\rho$, $R := \alpha g_a a^2 \beta/2\Omega\eta$, $E := \nu/2\Omega a^2$ and $q := \kappa/\eta$ are the magnetic Ekman, Elsasser, a modified Rayleigh and Roberts numbers. On the boundary the velocity satisfies the no-slip condition, $\mathbf{v} = \mathbf{0}$, the temperature vanishes, $\Theta = 0$, and the magnetic field matches to a potential field vanishing at infinity, equations (34)–(36). Thus

$$\partial_r S_n^m + (n+1)S_n^m = T_n^m = s_n^m = \partial_r s_n^m = t_n^m = \Theta_n^m = 0, \text{ at } r = 1.$$

TABLE 3: Critical Rayleigh number for the magneto-convection model.

N_n	N_r	R	$\Im\gamma$
10	50	110.03291	-43.14997
10	20	113.3	-42.35 [2]
20	20	115.0	-43.25 [4]

In the test model [2, 4] the basic state magnetic field is purely toroidal, $\mathbf{B}_0 = \sin\theta\,\mathbf{1}_\phi$, and the non-zero basic state vector spherical harmonic coefficients are $B_{1,1}^0 = -i\sqrt{2/3}\,r$ and $J_{1,1}^0 = 2$. The critical ($\Re\gamma = 0$) modified Rayleigh number R is shown in Table 3 for $\Lambda = 0.5$ and $m = 2$, and compared with less general methods [2] and [4]. The former [2] used a Galerkin method in radius and angle and was restricted only to the basic state given above; the latter [4] used a less accurate method, namely finite-differences in both r and θ with the ϕ-dependence separated out.

5   Concluding remarks

A hybrid spectral code for the linearised magnetohydrodynamic equations in a sphere, which uses vector spherical harmonics and toroidal-poloidal fields, is described. The basic state is axisymmetric. Good agreement is obtained with published results for three model problems.

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