Stability analysis from fourth order evolution equation for counter-propagating gravity wave packets in the presence of wind flowing over water

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Abstract

Fourth order nonlinear evolution equations are derived for two counter-propagating surface gravity wave packets in deep water in the presence of wind flowing over water. The resulting equations are asymptotically exact and nonlocal. Stability analysis is made for a uniform standing surface gravity wave train for longitudinal perturbation on the basis of these equations. Graphs are plotted for maximum growth rate of instability and for wave number at marginal stability against wave steepness for some different values of dimensionless wind velocity. Significant deviations are noticed between the results obtained from third order and fourth order nonlinear evolution equations. This paper has an application in rough waves.

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Contents

1 Introduction

One approach to studying the stability of finite amplitude surface gravity waves in deep water is through the application of the lowest order nonlinear evolution equation, which is the nonlinear Schrödinger equation. Zakharov's [21] finite amplitude wave trains to be subjected to modulational perturbations in two horizontal directions both along and perpendicular to the direction of the wave train. Davey and Stewartson [4] extended of this to water of finite depth. Further extensions of this were made by Djordjevic and Redekopp [9] to include capillarity and by Das [3] to include density stratification.

For small steepness, $ka < 0.1$, the predictions from the nonlinear Schr ödinger equation, when compared with Longuet-Higgin's [16, 17] exact results, are fairly accurate. Here k is the wavenumber and a is the amplitude of the wave.

But for steepness $ka > 0.15$ the predictions from the nonlinear Schrödinger equation do not agree with the exact results of Longuet-Higgins [16, 17]. Dysthe [11] has shown that a surprising improvement on these results relating to stability of a finite amplitude wave can be attained by extending the perturbation analysis one step further, that is adding the order ϵ^4 term in the nonlinear Schrödinger equation.

From this fourth-order evolution equation Janssen [15] has elaborated on the Dysthe [11] approach by investigating the effect of wave-induced flow on the long time behaviour of Benjamin–Feir [1] instability and also applied this equation to the homogeneous random field of gravity waves and obtained the nonlinear energy transfer function found by Dungey and Hui [10]. Stiassnie [19] shown that Zakharov's [21] integral equation yields the modified or fourth order nonlinear Schrödinger equation for the particular case of narrow spectrum. Hogan [14] considered the stability of a train of nonlinear capillary-gravity waves on the surface of an ideal fluid of infinite depth. He derived from the Zakharov's [21] equation under the assumption of a narrow band of waves and including the full from of interaction coefficient for capillary-gravity waves, an evolution equation for the wave envelope that is correct to fourth order in the wave steepness. Fourth order nonlinear evolution equation for deep water surface-gravity waves in different contexts and stability analysis made from them were derived by Dhar and Das [6, 7, 8], Debsarma and Das [5], Hara and Mei [12, 13], Bhattacharyya and Das [2]. The third order nonlinear evolution equations have been derived by Pierce and Knobloch [18] for two counter-propagating capillary gravity wave packets on the surface of water of finite depth. The resulting equations are asymptotically exact and nonlocal and generalize the equations derived by Davey and Stewartson [4] for unidirectional wave trains.

In the present paper fourth order nonlinear evolution equations are derived for two counter-propagating surface gravity wave packets in deep water in the presence of wind flowing over water. So this paper extends of the evolution equations derived by Pierce and Knobloch [18] for gravity waves to one order higher for an infinite depth water and in the presence of wind flowing over water.

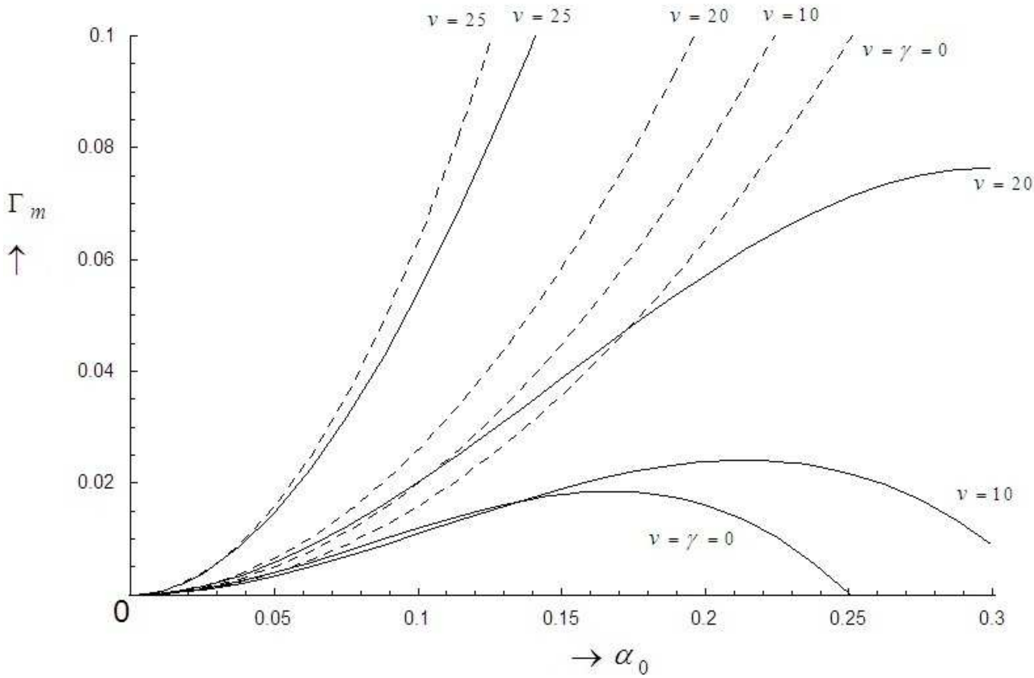

Figure 1: Maximum growth rate of instability Γ_m against wave steepness α_0 for some different values of dimensionless wind velocity ν. Here $\gamma = 0.00129$ for all the graphs except for the two with $\nu = 0$, $\gamma = 0$ written on the graph: ——, fourth order results; - - - -, third order results.

The evolution equations (35) and (36) remains valid when the wind velocity is less than a critical velocity. This critical velocity is defined by the fact that a wave becomes linearly unstable if the wind velocity exceeds this critical velocity. On the basis of these evolution equations stability analysis is investigated for a uniform standing surface gravity wave train with respect to longitudinal perturbation. The instability condition (58) is obtained and expressions for the maximum growth rate of instability and the wave number at marginal stability (59) are derived.

Figures 1–4 plot for maximum growth rate of instability and for wave number

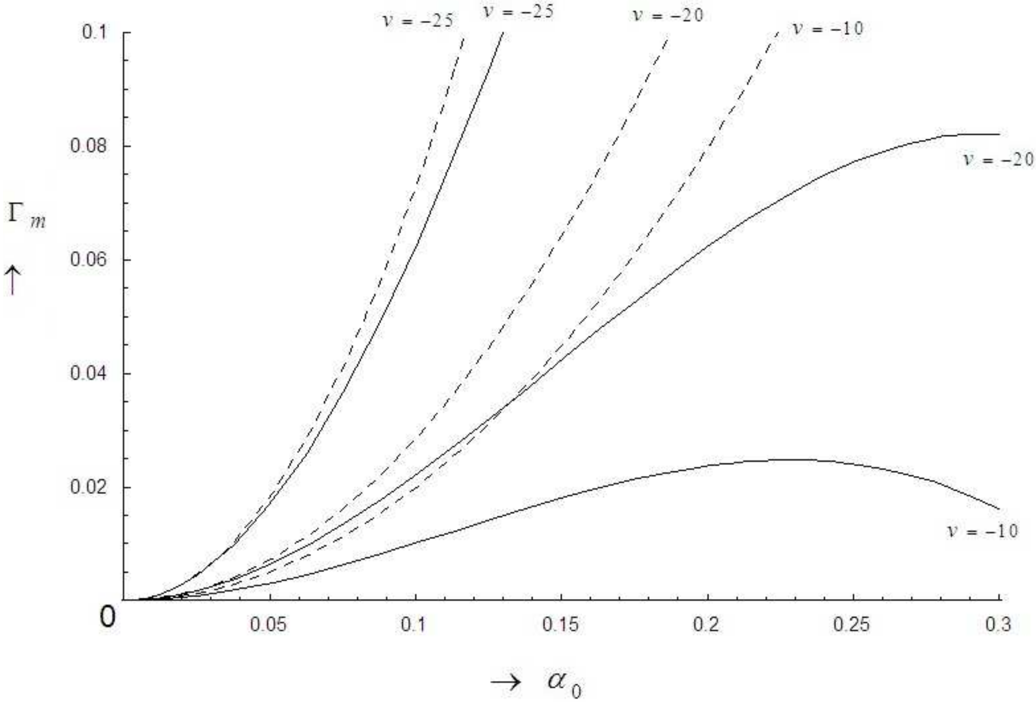

Figure 2: Maximum growth rate of instability Γ_m against wave steepness α_0 for some different values of dimensionless wind velocity ν. Here $\gamma = 0.00129$: ——, fourth order results; - - - -, third order results.

at marginal stability against wave steepness for some different values of dimensionless wind velocity. In the fourth order analysis for waves with sufficiently small wave numbers the maximum growth rate of instability first increases with the increase of wave steepness and then it decreases with the increase of wave steepness and finally vanishes at some critical value of wave steepness beyond which there is no instability, while in the third order analysis the maximum growth rate of instability increases steadily with the increase of wave steepness. The growth rate is found to be appreciably much higher for dimensionless wind velocity approaching its critical value. The wave number

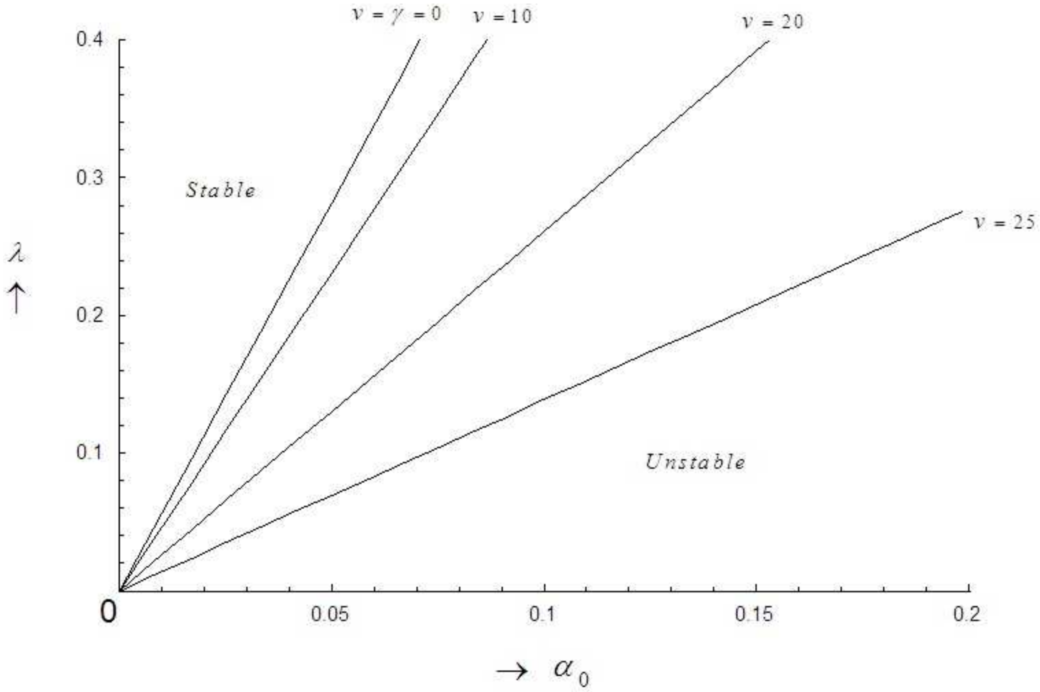

Figure 3: Wave number λ at marginal stability against wave steepness α_0 for some different values of dimensionless wind velocity ν. Here $\gamma = 0.00129$ for all the graphs except for the one with $\nu = 0$, $\gamma = 0$ written on the graph.

at marginal stability has also been plotted against wave steepness for some different values of dimensionless wind velocity.

2 Basic equations

We take the common horizontal interface between water and air in the undisturbed state as $z = 0$ plane. In the undisturbed state air flows over water with a velocity u in a direction that is taken as the x-axis. We take

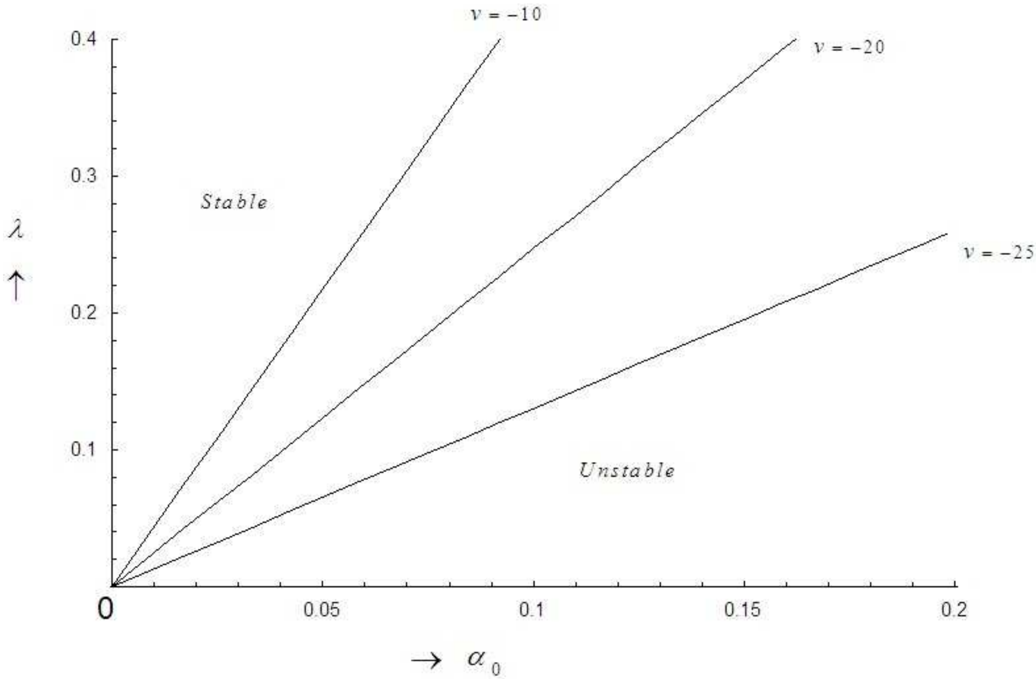

Figure 4: Wave number λ at marginal stability against wave steepness α_0 for some different values of dimensionless wind velocity ν. Here $\gamma = 0.00129$ for all the graphs.

$z = \zeta(x, y, t)$ as the equation of the common interface at any time t in the perturbed state. Let ρ and ρ' be the densities of water and air respectively. We introduce the dimensionless quantities $\widetilde{\phi}$, $\widetilde{\phi}'$, $\widetilde{\zeta}$, $(\widetilde{x}, \widetilde{y}, \widetilde{z})$, $\widetilde{t}$, $\widetilde{\nu}$ and $\widetilde{\gamma}$ which are, respectively, the perturbed velocity potential in water, perturbed velocity potential in air, surface elevation of the water-air interface, space coordinates, time, air flow velocity, and the ratio of the densities of air to water.

These dimensionless quantities are related to the corresponding dimensional quantities by the following relations

$$\widetilde{\phi} = \sqrt{k_0^3/g}\,\phi, \quad \widetilde{\phi}' = \sqrt{k_0^3/g}\,\phi', \quad (\widetilde{x}, \widetilde{y}, \widetilde{z}) = (k_0 x, k_0 y, k_0 z),$$

$$\tilde{\zeta} = k_0 \zeta, \quad \tilde{t} = \omega t, \quad \tilde{\nu} = \sqrt{k_0/g}\, u, \quad \tilde{\gamma} = \rho'/\rho,$$

where k_0 is some characteristic wave number, g is the acceleration due to gravity. Here after all the quantities will be written in their dimensionless form with their over tilde ($\tilde{}$) dropped.

The perturbed velocity potentials ϕ and ϕ' satisfy the Laplace equations

$$\nabla^2 \phi = 0 \quad \text{in} \quad -\infty < z < \zeta\,, \tag{1}$$

$$\nabla^2 \phi' = 0 \quad \text{in} \quad \zeta < z < \infty\,. \tag{2}$$

The kinematic boundary condition for water is

$$\frac{\partial \phi}{\partial z} - \frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y} \quad \text{when } z = \zeta\,, \tag{3}$$

which gives a necessary condition for equality of water velocity at the interface normal to it to the normal velocity of the interface. The similar condition for air is

$$\frac{\partial \phi'}{\partial z} - \frac{\partial \zeta}{\partial t} - \nu\frac{\partial \zeta}{\partial x} = \frac{\partial \phi'}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi'}{\partial y}\frac{\partial \zeta}{\partial y} \quad \text{when } z = \zeta\,. \tag{4}$$

The condition of continuity of pressure at the interface gives

$$\left\{\frac{\partial \phi}{\partial t} - \gamma\frac{\partial \phi'}{\partial t} + (1-\gamma)\,\zeta - \gamma\nu\frac{\partial \phi'}{\partial x}\right\} + \frac{1}{2}\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right\}$$
$$- \frac{\gamma}{2}\left\{\left(\frac{\partial \phi'}{\partial x}\right)^2 + \left(\frac{\partial \phi'}{\partial y}\right)^2 + \left(\frac{\partial \phi'}{\partial z}\right)^2\right\} = 0 \quad \text{when } z = \zeta\,. \tag{5}$$

Finally ϕ and ϕ' should satisfy the following boundary conditions at infinity

$$\frac{\partial \phi}{\partial z} \to 0 \quad \text{when } z \to -\infty\,, \tag{6}$$

$$\frac{\partial \phi'}{\partial z} \to 0 \quad \text{when } z \to +\infty\,. \tag{7}$$

Since the disturbance is assumed to be a progressive wave we look for solutions of the equations (1)–(7) in the form

$$P = P_{00} + \sum_{m,n=-\infty}^{\infty} \{P_{mn} \exp[i(m\psi_1 + n\psi_2)] + P^*_{mn} \exp[-i(m\psi_1 + n\psi_2)]\}\,, \tag{8}$$

where P stands for ϕ, ϕ' and ζ; $\psi_1 = kx - \omega t$, $\psi_2 = kx + \omega t$. In the summation on the right hand side of equation (8), $(m, n) \neq (0, 0)$. Here ϕ_{00}, ϕ_{mn}, ϕ^*_{mn}, ϕ'_{00}, ϕ'_{mn} and ϕ'^*_{mn} are functions of z, $x_1 = \epsilon x$, $y_1 = \epsilon y$, $t_1 = \epsilon t$; ζ_{00}, ζ_{mn} and ζ^*_{mn} are functions of x_1, y_1 and t_1. The small parameter ϵ measures the weakness of wave steepness, which is the product of wave amplitude and wave number. The sign * denotes complex conjugate.

The linear dispersion relation determining frequency ω is

$$(1+\gamma)\,\omega^2 - 2\gamma\omega\nu + \gamma\nu^2 - (1-\gamma) = 0\,. \tag{9}$$

This equation gives two values

$$\omega_\pm = \left(\gamma\nu \pm \sqrt{1-\gamma^2-\gamma\nu^2}\right)/\,(1+\gamma)\,, \tag{10}$$

which corresponds to two modes and we designate these two modes as $+$ and $-$ modes. The positive mode moves in the positive direction of the x-axis with a frequency $\left(\sqrt{1-\gamma^2-\gamma\nu^2}+\gamma\nu\right)/\,(1+\gamma)$ while the negative mode moves in the negative direction of the x-axis with a frequency $\left(\sqrt{1-\gamma^2-\gamma\nu^2}-\gamma\nu\right)/\,(1+\gamma)$. If ν is replaced by $-\nu$, then the frequency of the positive mode becomes equal to the frequency of the negative mode. So the results for the negative mode can be obtained from those for the positive mode by replacing ν by $-\nu$. Therefore we have made a nonlinear analysis for the positive mode only and then we obtained the results for the negative mode by replacing ν by $-\nu$.

From the expression (10) for $\omega_\pm$ we find that for linear stability, velocity ν should satisfy the condition

$$|\nu| < \sqrt{(1-\gamma^2)/\gamma}\,. \tag{11}$$

Thus our present analysis will remain valid as long as the dimensionless flow velocity of the wind becomes less than the critical value $\sqrt{(1-\gamma^2)/\gamma}$. For air flowing over water $\gamma = 0.00129$ and this critical value becomes 27.84. The corresponding dimensional value of critical wind velocity is 872.05 cm/sec.

3 Derivation of evolution equations

On substituting the expansions (8) in equations (1), (2), (6) and (7), and then equating the coefficients of $\exp[i(m\psi_1 + n\psi_2)]$ for $(m,n) = (1,0)$, $(0,1)$, $(2,0)$, $(0,2)$, $(1,1)$ and $(-1,1)$ we get the following equations:

$$\left(\frac{\partial^2}{\partial z^2} - \Delta_{mn}^2\right)\phi_{mn} = 0\,, \tag{12}$$

$$\left(\frac{\partial^2}{\partial z^2} - \Delta_{mn}^2\right)\phi'_{mn} = 0\,, \tag{13}$$

$$\frac{\partial \phi_{mn}}{\partial z} \to 0 \quad \text{as} \quad z \to -\infty\,, \tag{14}$$

$$\frac{\partial \phi'_{mn}}{\partial z} \to 0 \quad \text{as} \quad z \to +\infty\,, \tag{15}$$

$$\text{where} \quad \Delta_{mn}^2 = \left\{(m+n) - i\epsilon\frac{\partial}{\partial x_1}\right\}^2 - \epsilon^2\frac{\partial^2}{\partial y_1^2}\,. \tag{16}$$

The solutions of equations (12) and (13) satisfying boundary conditions (14) and (15) respectively put in the following forms

$$\phi_{mn} = \exp\left(\Delta_{mn} z\right) A_{mn}\,, \tag{17}$$

$$\phi'_{mn} = \exp\left(-\Delta_{mn} z\right) A'_{mn}\,, \tag{18}$$

where A_{mn} and A'_{mn} are functions of x_1, y_1 and t_1. For the sake of convenience we take the Fourier transforms of equations (1), (2), (6) and (7) for $(m,n) = (0,0)$. The solutions of these transformed equations become

$$\bar{\phi}_{00} = \exp\left(|\bar{k}|z\right)\bar{A}_{00}\,, \tag{19}$$

$$\bar{\phi}'_{00} = \exp\left(-|\bar{k}|z\right)\bar{A}'_{00}\,, \tag{20}$$

where $\bar{\phi}_{00}$ and $\bar{\phi}'_{00}$ are Fourier transforms of ϕ_{00} and ϕ'_{00} respectively, defined by

$$\left(\bar{\phi}_{00}, \bar{\phi}'_{00}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\phi_{00}, \phi'_{00}\right)\exp\left[i(\bar{k}_x x_1 + \bar{k}_y y_1 - \bar{\omega} t_1)\right]\,dx_1\,dy_1\,dt_1 \tag{21}$$

where $\bar{k}^2 = \left(\bar{k}_x^2 + \bar{k}_y^2\right)$, and where $\bar{A}_{00}$ and $\bar{A}'_{00}$ are functions of $\bar{k}_x$, $\bar{k}_y$ and $\bar{\omega}$.

On substituting the expansions (8) in the Taylor expanded forms of equations (3)–(5) about $z = 0$ and then equating the coefficients of $\exp[i(m\psi_1 + n\psi_2)]$ for $(m, n) = (1, 0)$, $(0, 1)$, $(2, 0)$, $(0, 2)$, $(1, 1)$, $(-1, 1)$ and $(0, 0)$ on both sides, we get

$$\left(\frac{\partial \phi_{mn}}{\partial z}\right)_{z=0} + i\left\{(m-n)\omega + i\epsilon\frac{\partial}{\partial t_1}\right\}\zeta_{mn} = a_{mn}\,, \tag{22}$$

$$\left(\frac{\partial \phi'_{mn}}{\partial z}\right)_{z=0} + i\left\{(m-n)\omega + i\epsilon\frac{\partial}{\partial t_1}\right\}\zeta_{mn} - i\nu\left\{(m+n) - i\epsilon\frac{\partial}{\partial x_1}\right\}\zeta_{mn} = b_{mn}\,, \tag{23}$$

$$-i\left\{(m-n)\omega + i\epsilon\frac{\partial}{\partial t_1}\right\}(\phi_{mn})_{z=0} + i\gamma\left\{(m-n)\omega + i\epsilon\frac{\partial}{\partial t_1}\right\}(\phi'_{mn})_{z=0} + (1-\gamma)\zeta_{mn} - i\gamma\nu\left\{(m+n) - i\epsilon\frac{\partial}{\partial x_1}\right\}(\phi'_{mn})_{z=0} = c_{mn}\,. \tag{24}$$

where a_{mn}, b_{mn} and c_{mn} are contributions from unexpanded nonlinear terms and $(\ )_{z=0}$ implies the value of the quantity inside parentheses at $z = 0$. Now for the above seven values of (m, n) we obtain seven sets of equations, in which we substitute the solutions for ϕ_{mn} and ϕ'_{mn} given by (17)–(20). For the sake of convenience we take the Fourier transforms of the set of equations corresponding to $(m, n) = (0, 0)$. The sets of equations corresponding to $(m, n) \in \{(1, 0), (0, 1)\}$, $(m, n) \in \{(2, 0), (0, 2), (1, 1), (-1, 1)\}$, $(m, n) = (0, 0)$ will be called, respectively, the first, second and third sets.

To solve the above three sets of equations we make the following perturbation expansions for the quantities A_{mn}, A'_{mn} and ζ_{mn} for the above values of (m, n):

$$E_{mn} = \begin{cases} \sum_{p=1}^{\infty} \epsilon^p E_{mn}^{(p)} & (m,n) = (1,0),(0,1), \\ \sum_{p=2}^{\infty} \epsilon^p E_{mn}^{(p)} & (m,n) = (2,0),(0,2),(1,1),(-1,1),(0,0), \end{cases} \quad (25)$$

where E_{mn} stands for A_{mn}, A'_{mn} and ζ_{mn}.

On substituting the expansions (25) in the above three sets of equations and then equating coefficients of various powers of ϵ on both sides, we obtain a sequence of equations. From the first order (that is, lowest order) and second order equations corresponding to (22) and (23) of the first set of equations we obtain solutions for $A_{10}^{(1)}$, $A_{10}^{\prime(1)}$, $A_{10}^{(2)}$, $A_{10}^{\prime(2)}$ and $A_{01}^{(1)}$, $A_{01}^{\prime(1)}$, $A_{01}^{(2)}$, $A_{01}^{\prime(2)}$ respectively. Next, from the second order and third order equations corresponding to (22), (23) and (24) of the second set of equations, we obtain solutions for $(A_{20}^{(2)}, A_{20}^{\prime(2)}, \zeta_{20}^{(2)}, A_{20}^{(3)}, A_{20}^{\prime(3)}, \zeta_{20}^{(3)})$, $(A_{02}^{(2)}, A_{02}^{\prime(2)}, \zeta_{02}^{(2)}, A_{02}^{(3)}, A_{02}^{\prime(3)}, \zeta_{02}^{(3)})$, $(A_{11}^{(2)}, A_{11}^{\prime(2)}, \zeta_{11}^{(2)}, A_{11}^{(3)}, A_{11}^{\prime(3)}, \zeta_{11}^{(3)})$, $(A_{-11}^{(2)}, A_{-11}^{\prime(2)}, \zeta_{-11}^{(2)}, A_{-11}^{(3)}, A_{-11}^{\prime(3)}, \zeta_{-11}^{(3)})$ respectively. Finally, from the second order equations corresponding to equations (22), (23) and (24) of the third set of equations we obtain solutions for $(A_{00}^{(2)}, A_{00}^{\prime(2)}, \zeta_{00}^{(2)})$ and from the third order equation corresponding to (24) of the third set of equations we obtain a solution for $\zeta_{00}^{(3)}$. Following Pierce and Knobloch [18] we use the following transformations of all perturbed quantities in slow space coordinates and time

$$\xi_+ = x_1 - c_g t_1, \quad \xi_- = x_1 + c_g t_1, \quad \eta = y_1, \quad \tau_1 = \epsilon t_1, \quad \tau_2 = \epsilon^2 t_1, \quad (26)$$

where $c_g = (d\omega/dk)_{k=1}$ is the group velocity. As we are going to derive evolution equation correct up to $O(\epsilon^4)$ which is one order higher than the evolution equation in the lowest order, we introduce one more slow time variable τ_2 following Weissman [20]. The equations corresponding to (24) for $(m, n) = (1, 0)$ and $(0, 1)$ of the first set of equations, which has not been used in obtaining the above perturbation solutions put in the following convenient forms after eliminating $A_{mn}^{(p)}$ and $A'^{(p)}_{mn}$, the details of which are shown in Appendix A.

$$\left[\omega_1^2 + \gamma\,(\omega_1 - \nu k_1)^2 - (1-\gamma)\,\Delta_{10}\right]\zeta_{10} = -i\omega_1 a_{10} - i\gamma\,(\omega_1 - \nu k_1)\,b_{10} - \Delta_{10} c_{10}\,, \tag{27}$$

$$\left[\omega_1^2 + \gamma\,(\omega_1 - \nu k_1)^2 - (1-\gamma)\,\Delta_{01}\right]\zeta_{01} = -i\omega_1 a_{01} - i\gamma\,(\omega_1 - \nu k_1)\,b_{01} - \Delta_{01} c_{01}\,, \tag{28}$$

where a_{10}, b_{10}, c_{10}, a_{01}, b_{01} and c_{01} are contributions from nonlinear terms.

From equations (27) and (28) we get the following equations in three successive orders starting from the lowest order two.

$O(\epsilon^2)$

$$\frac{\zeta_{10}^{(1)}}{\partial\xi_-} = 0\,, \tag{29}$$

$$\frac{\zeta_{01}^{(1)}}{\partial\xi_+} = 0\,. \tag{30}$$

These two equations show that $\zeta_{10}^{(1)}$ and $\zeta_{01}^{(1)}$ are independent of ξ_- and ξ_+ respectively.

$O(\epsilon^3)$

$$i\frac{\partial\zeta_{10}^{(1)}}{\partial\tau_1} + i\gamma_0\frac{\partial\zeta_{10}^{(2)}}{\partial\xi_-} + \gamma_1\frac{\partial^2\zeta_{10}^{(1)}}{\partial\xi_+^2} + \gamma_2\frac{\partial^2\zeta_{10}^{(1)}}{\partial\eta^2} = \delta_1 {\zeta_{10}^{(1)}}^2 {\zeta_{10}^{(1)}}^* + \delta_2 \zeta_{10}^{(1)}\zeta_{01}^{(1)}{\zeta_{01}^{(1)}}^*, \tag{31}$$

$$-i\frac{\partial\zeta_{01}^{(1)}}{\partial\tau_1} + i\gamma_0\frac{\partial\zeta_{01}^{(2)}}{\partial\xi_+} + \gamma_1\frac{\partial^2\zeta_{01}^{(1)}}{\partial\xi_-^2} + \gamma_2\frac{\partial^2\zeta_{01}^{(1)}}{\partial\eta^2} = \delta_1 {\zeta_{01}^{(1)}}^2 {\zeta_{01}^{(1)}}^* + \delta_2 \zeta_{01}^{(1)}\zeta_{10}^{(1)}{\zeta_{10}^{(1)}}^*. \tag{32}$$

$O(\epsilon^4)$

$$
\begin{aligned}
&i\frac{\partial\zeta_{10}^{(2)}}{\partial\tau_1}+i\frac{\partial\zeta_{10}^{(1)}}{\partial\tau_2}+\gamma_1\left(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\xi_-^2}\right)\zeta_{10}^{(2)}+\gamma_2\frac{\partial^2\zeta_{10}^{(2)}}{\partial\eta^2}+\gamma_3\frac{\partial^2\zeta_{10}^{(2)}}{\partial\xi_+\partial\xi_-}\\
&\quad+i\gamma_4\frac{\partial^3\zeta_{10}^{(1)}}{\partial\xi_+^3}+i\gamma_5\frac{\partial^3\zeta_{10}^{(1)}}{\partial\xi_+\partial\eta^2}=\delta_1{\zeta_{10}^{(1)}}^2{\zeta_{10}^{(2)}}^*+2\delta_1\zeta_{10}^{(1)}\zeta_{10}^{(2)}{\zeta_{10}^{(1)}}^*\\
&\quad+\delta_2\zeta_{10}^{(1)}\zeta_{01}^{(1)}{\zeta_{01}^{(2)}}^*+\delta_2\zeta_{10}^{(1)}\zeta_{01}^{(2)}{\zeta_{01}^{(1)}}^*+\delta_2\zeta_{10}^{(2)}\zeta_{01}^{(1)}{\zeta_{01}^{(1)}}^*\\
&\quad+i\delta_3\zeta_{10}^{(1)}{\zeta_{10}^{(1)}}^*\frac{\partial\zeta_{10}^{(1)}}{\partial\xi_+}+i\delta_4{\zeta_{10}^{(1)}}^2\frac{\partial{\zeta_{10}^{(1)}}^*}{\partial\xi_+}+i\delta_5\zeta_{10}^{(1)}\zeta_{01}^{(1)}\frac{\partial{\zeta_{01}^{(1)}}^*}{\partial\xi_-}\\
&\quad+i\delta_6\zeta_{10}^{(1)}{\zeta_{01}^{(1)}}^*\frac{\partial\zeta_{01}^{(1)}}{\partial\xi_-}+i\delta_7\zeta_{01}^{(1)}{\zeta_{01}^{(1)}}^*\frac{\partial\zeta_{10}^{(1)}}{\partial\xi_+}\\
&\quad+2\zeta_{10}^{(1)}\left[\frac{\partial}{\partial\xi_+}F^{-1}\left\{\frac{1}{\bar{\bar{k}}}F\frac{\partial}{\partial\xi_+}(|\zeta_{10}^{(1)}|^2)\right\}-\frac{\partial}{\partial\xi_-}F^{-1}\left\{\frac{1}{\bar{\bar{k}}}F\frac{\partial}{\partial\xi_-}(|\zeta_{01}^{(1)}|^2)\right\}\right],
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
&-i\frac{\partial\zeta_{01}^{(2)}}{\partial\tau_1}+i\frac{\partial\zeta_{01}^{(1)}}{\partial\tau_2}+\gamma_1\left(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\xi_-^2}\right)\zeta_{01}^{(2)}+\gamma_2\frac{\partial^2\zeta_{01}^{(2)}}{\partial\eta^2}+\gamma_3\frac{\partial^2\zeta_{01}^{(2)}}{\partial\xi_+\partial\xi_-}\\
&\quad+i\gamma_4\frac{\partial^3\zeta_{01}^{(1)}}{\partial\xi_-^3}+i\gamma_5\frac{\partial^3\zeta_{01}^{(1)}}{\partial\xi_-\partial\eta^2}=\delta_1{\zeta_{01}^{(1)}}^2{\zeta_{01}^{(2)}}^*+2\delta_1\zeta_{01}^{(1)}\zeta_{01}^{(2)}{\zeta_{01}^{(1)}}^*\\
&\quad+\delta_2\zeta_{01}^{(1)}\zeta_{10}^{(1)}{\zeta_{10}^{(2)}}^*+\delta_2\zeta_{01}^{(1)}\zeta_{10}^{(2)}{\zeta_{10}^{(1)}}^*+\delta_2\zeta_{01}^{(2)}\zeta_{10}^{(1)}{\zeta_{10}^{(1)}}^*\\
&\quad+i\delta_3\zeta_{01}^{(1)}{\zeta_{01}^{(1)}}^*\frac{\partial\zeta_{01}^{(1)}}{\partial\xi_-}+i\delta_4{\zeta_{01}^{(1)}}^2\frac{\partial{\zeta_{01}^{(1)}}^*}{\partial\xi_-}+i\delta_5\zeta_{01}^{(1)}\zeta_{10}^{(1)}\frac{\partial{\zeta_{10}^{(1)}}^*}{\partial\xi_+}\\
&\quad+i\delta_6\zeta_{01}^{(1)}{\zeta_{10}^{(1)}}^*\frac{\partial\zeta_{10}^{(1)}}{\partial\xi_+}+i\delta_7\zeta_{10}^{(1)}{\zeta_{10}^{(1)}}^*\frac{\partial\zeta_{01}^{(1)}}{\partial\xi_-}\\
&\quad+2\zeta_{01}^{(1)}\left[\frac{\partial}{\partial\xi_-}F^{-1}\left\{\frac{1}{\bar{\bar{k}}}F\frac{\partial}{\partial\xi_-}(|\zeta_{01}^{(1)}|^2)\right\}-\frac{\partial}{\partial\xi_+}F^{-1}\left\{\frac{1}{\bar{\bar{k}}}F\frac{\partial}{\partial\xi_+}(|\zeta_{10}^{(1)}|^2)\right\}\right],
\end{aligned}
\tag{34}
$$

where $F^{-1}(\ )$ is the inverse Fourier transform of the quantity inside the parentheses.

Now arranging different terms of equations (31)–(34) we obtain the following fourth order nonlinear evolution equations for two counter-propagating waves.

$$
\begin{aligned}
& i\frac{\partial \zeta_{10}^{(1)}}{\partial \tau_1} + i\gamma_0 \frac{\partial \zeta_{10}^{(2)}}{\partial \xi_-} + \gamma_1 \left(\frac{\partial^2}{\partial \xi_+^2} + \frac{\partial^2}{\partial \xi_-^2} \right) \zeta_{10}^{(1)} + \gamma_2 \frac{\partial^2 \zeta_{10}^{(1)}}{\partial \eta^2} \\
& \quad + \epsilon \left[i\frac{\partial \zeta_{10}^{(2)}}{\partial \tau_1} + i\frac{\partial \zeta_{10}^{(1)}}{\partial \tau_2} + \gamma_1 \left(\frac{\partial^2}{\partial \xi_+^2} + \frac{\partial^2}{\partial \xi_-^2} \right) \zeta_{10}^{(2)} + \gamma_2 \frac{\partial^2 \zeta_{10}^{(2)}}{\partial \eta^2} + \gamma_3 \frac{\partial^2 \zeta_{10}^{(2)}}{\partial \xi_+ \partial \xi_-} \right. \\
& \quad \left. + i\gamma_4 \frac{\partial^3 \zeta_{10}^{(1)}}{\partial \xi_+^3} + i\gamma_5 \frac{\partial^3 \zeta_{10}^{(1)}}{\partial \xi_+ \partial \eta^2} \right] = \delta_1 |\zeta_{10}^{(1)}|^2 \zeta_{10}^{(1)} + \delta_2 |\zeta_{01}^{(1)}|^2 \zeta_{10}^{(1)} \\
& \quad + \epsilon \left[\delta_1 \zeta_{10}^{(1)^2} \zeta_{10}^{(2)^*} + 2\delta_1 |\zeta_{10}^{(1)}|^2 \zeta_{10}^{(2)} + \delta_2 \zeta_{10}^{(1)} \zeta_{01}^{(1)} \zeta_{01}^{(2)^*} + \delta_2 \zeta_{10}^{(1)} \zeta_{01}^{(1)^*} \zeta_{01}^{(2)} \right. \\
& \quad + \delta_2 \zeta_{10}^{(2)} |\zeta_{01}^{(1)}|^2 + i\delta_3 |\zeta_{10}^{(1)}|^2 \frac{\partial \zeta_{10}^{(1)}}{\partial \xi_+} + i\delta_4 \zeta_{10}^{(1)^2} \frac{\partial \zeta_{10}^{(1)^*}}{\partial \xi_+} + i\delta_5 \zeta_{10}^{(1)} \zeta_{01}^{(1)} \frac{\partial \zeta_{01}^{(1)^*}}{\partial \xi_-} \\
& \quad + i\delta_6 \zeta_{10}^{(1)} \zeta_{01}^{(1)^*} \frac{\partial \zeta_{01}^{(1)}}{\partial \xi_-} + i\delta_7 |\zeta_{01}^{(1)}|^2 \frac{\partial \zeta_{10}^{(1)}}{\partial \xi_+} \\
& \quad \left. + 2\zeta_{10}^{(1)} \left\{ H \frac{\partial}{\partial \xi_+} (|\zeta_{10}^{(1)}|^2)| - H \frac{\partial}{\partial \xi_-} (|\zeta_{01}^{(1)}|^2) \right\} \right] \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
& -i\frac{\partial \zeta_{01}^{(1)}}{\partial \tau_1} + i\gamma_0 \frac{\partial \zeta_{01}^{(2)}}{\partial \xi_+} + \gamma_1 \left(\frac{\partial^2}{\partial \xi_+^2} + \frac{\partial^2}{\partial \xi_-^2} \right) \zeta_{01}^{(1)} + \gamma_2 \frac{\partial^2 \zeta_{01}^{(1)}}{\partial \eta^2} + \epsilon \left[-i\frac{\partial \zeta_{01}^{(2)}}{\partial \tau_1} \right. \\
& \quad - i\frac{\partial \zeta_{01}^{(1)}}{\partial \tau_2} + \gamma_1 \left(\frac{\partial^2}{\partial \xi_+^2} + \frac{\partial^2}{\partial \xi_-^2} \right) \zeta_{01}^{(2)} + \gamma_2 \frac{\partial^2 \zeta_{01}^{(2)}}{\partial \eta^2} + \gamma_3 \frac{\partial^2 \zeta_{01}^{(2)}}{\partial \xi_+ \partial \xi_-} + i\gamma_4 \frac{\partial^3 \zeta_{01}^{(1)}}{\partial \xi_-^3} \\
& \quad \left. + i\gamma_5 \frac{\partial^3 \zeta_{01}^{(1)}}{\partial \xi_- \partial \eta^2} \right] = \delta_1 |\zeta_{01}^{(1)}|^2 \zeta_{01}^{(1)} + \delta_2 |\zeta_{10}^{(1)}|^2 \zeta_{01}^{(1)} + \epsilon \left[\delta_1 \zeta_{01}^{(1)^2} \zeta_{01}^{(2)^*} \right. \\
& \quad + 2\delta_1 |\zeta_{01}^{(1)}|^2 \zeta_{01}^{(2)} + \delta_2 \zeta_{01}^{(1)} \zeta_{10}^{(1)} \zeta_{10}^{(2)^*} + \delta_2 \zeta_{01}^{(1)} \zeta_{10}^{(1)^*} \zeta_{10}^{(2)} + \delta_2 \zeta_{01}^{(2)} |\zeta_{10}^{(1)}|^2 \\
& \quad + i\delta_3 |\zeta_{01}^{(1)}|^2 \frac{\partial \zeta_{01}^{(1)}}{\partial \xi_-} + i\delta_4 \zeta_{01}^{(1)^2} \frac{\partial \zeta_{01}^{(1)^*}}{\partial \xi_-} + i\delta_5 \zeta_{01}^{(1)} \zeta_{10}^{(1)} \frac{\partial \zeta_{10}^{(1)^*}}{\partial \xi_+} + i\delta_6 \zeta_{01}^{(1)} \zeta_{10}^{(1)^*} \frac{\partial \zeta_{10}^{(1)}}{\partial \xi_+}
\end{aligned}
$$

$$+ i\delta_7|\zeta_{10}^{(1)}|^2\frac{\partial\zeta_{01}^{(1)}}{\partial\xi_-} + 2\zeta_{01}^{(1)}\left\{H\frac{\partial}{\partial\xi_-}\left(|\zeta_{01}^{(1)}|^2\right) - H\frac{\partial}{\partial\xi_+}\left(|\zeta_{10}^{(1)}|^2\right)\right\}\Bigg] \tag{36}$$

where the Hilbert transform operator

$$H\psi(\xi,\eta,\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\xi'-\xi)\psi(\xi',\eta',\tau)}{[(\xi'-\xi)^2+(\eta'-\eta)^2]^{3/2}}\,d\xi'\,d\eta'$$

and Appendix B gives the coefficients γ_i and δ_i.

In equation (35), if we restrict to the nonlinear evolution of unidirectional wave train propagating in the positive direction of x-axis, that is, if we set $\zeta_{01} = 0$ and assume that ζ_{10} is independent of ξ_-, then we recover the fourth-order nonlinear evolution equation for a gravity waves in the presence of wind flowing over water. This reduced equation is found to be same as equation (34) of Dhar and Das [6] if we set $\zeta = \zeta_{10}^{(1)} + \epsilon\zeta_{10}^{(2)}$ and $\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\tau_1} + \epsilon\frac{\partial}{\partial\tau_2}$ in equation (34). This reduced equation for $\nu = 0\,,\ \gamma = 0$ has also been verified by us to be equivalent to equation (2) of Janssen [15].

As each of the left and right propagating waves sees the counter-propagating wave only through its mean square amplitude, the nonlocal mean field equations suitable for stability analysis obtained from (35) and (36) is obtained by applying the averaging procedure of Pierce and Knobloch [18]. Therefore, following them we define the average of a function of two variables ξ_+ and ξ_- with respect to any one of these two variables by

$$\langle h\rangle_\pm = \frac{1}{p_\pm}\int_{-p_\pm/2}^{p_\pm/2} h\,d\xi_\pm\,, \tag{37}$$

where p_+ and p_- are the periods of the function h with respect to ξ_+ and ξ_- respectively. If h is not periodic, then by $\langle h\rangle_\pm$ we mean

$$\langle h\rangle_\pm = \int_{-\infty}^{\infty} h\ d\xi_\pm \tag{38}$$

provided the above integral exists.

Taking the average of equation (35) with respect to ξ_- over the period of ζ_{10} we get the following fourth order nonlocal mean-field equation for ζ_{10}:

$$i\frac{\partial\zeta_{10}^{(1)}}{\partial\tau_1}+\gamma_1\frac{\partial^2\zeta_{10}^{(1)}}{\partial\xi_+^2}+\gamma_2\frac{\partial^2\zeta_{10}^{(1)}}{\partial\eta^2}+\epsilon\left[i\frac{\partial\zeta_{10}^{(1)}}{\partial\tau_2}+i\frac{\partial\left\langle\zeta_{10}^{(2)}\right\rangle_-}{\partial\tau_1}+\gamma_1\frac{\partial^2\left\langle\zeta_{10}^{(2)}\right\rangle_-}{\partial\xi_+^2}\right.$$
$$\left.+\gamma_2\frac{\partial^2\left\langle\zeta_{10}^{(2)}\right\rangle_-}{\partial\eta^2}+i\gamma_4\frac{\partial^3\zeta_{10}^{(1)}}{\partial\xi_+^3}+i\gamma_5\frac{\partial^3\zeta_{10}^{(1)}}{\partial\xi_+\partial\eta^2}\right]=\delta_1|\zeta_{10}^{(1)}|^2\zeta_{10}^{(1)}+\delta_2|\zeta_{01}^{(1)^2}|\zeta_{10}^{(1)}$$
$$+\epsilon\left[\delta_1\zeta_{10}^{(1)^2}\left\langle\zeta_{10}^{(2)^*}\right\rangle_-+2\delta_1|\zeta_{10}^{(1)}|^2\left\langle\zeta_{10}^{(2)}\right\rangle_-+\delta_2\zeta_{10}^{(1)}\left\langle\zeta_{01}^{(1)}\zeta_{01}^{(2)^*}\right\rangle_-\right.$$
$$+\delta_2\zeta_{10}^{(1)}\left\langle\zeta_{01}^{(1)^*}\zeta_{01}^{(2)}\right\rangle_-+\delta_2\left\langle\zeta_{10}^{(2)}|\zeta_{01}^{(1)}|^2\right\rangle_-+i\delta_3|\zeta_{10}^{(1)}|^2\frac{\partial\zeta_{10}^{(1)}}{\partial\xi_+}+i\delta_4\zeta_{10}^{(1)^2}\frac{\partial\zeta_{10}^{(1)*}}{\partial\xi_+}$$
$$\left.+i\delta_7\left\langle|\zeta_{01}^{(1)}|^2\right\rangle_-\frac{\partial\zeta_{10}^{(1)}}{\partial\xi_+}+\delta_8\zeta_{10}^{(1)}+2\zeta_{10}^{(1)}H\frac{\partial}{\partial\xi_+}(|\zeta_{10}^{(1)}|^2)\right], \tag{39}$$

where

$$\delta_8=i\delta_5\left\langle\zeta_{01}^{(1)}\frac{\partial\zeta_{01}^{(1)^*}}{\partial\xi_-}\right\rangle_-+i\delta_6\left\langle\zeta_{01}^{(1)^*}\frac{\partial\zeta_{01}^{(1)}}{\partial\xi_-}\right\rangle_--2\left\langle H\frac{\partial}{\partial\xi_-}(|\zeta_{10}^{(1)}|^2)\right\rangle_-.$$

Similarly, taking the average of (36) with respect to ξ_+ over the period of ζ_{01} we get the following fourth order nonlocal mean-field evolution equation for ζ_{01}:

$$-i\frac{\partial\zeta_{01}^{(1)}}{\partial\tau_1}+\gamma_1\frac{\partial^2\zeta_{01}^{(2)}}{\partial\xi_-^2}+\gamma_2\frac{\partial^2\zeta_{01}^{(1)}}{\partial\eta^2}+\epsilon\left[-i\frac{\partial\zeta_{01}^{(1)}}{\partial\tau_2}-i\frac{\partial}{\partial\tau_1}\left\langle\zeta_{01}^{(2)}\right\rangle_+\right.$$
$$\left.+\gamma_1\frac{\partial^2}{\partial\xi_-^2}\left\langle\zeta_{01}^{(2)}\right\rangle_++\gamma_2\frac{\partial^2\left\langle\zeta_{01}^{(2)}\right\rangle_+}{\partial\eta^2}+i\gamma_4\frac{\partial^3\zeta_{01}^{(1)}}{\partial\xi_-^3}+i\gamma_5\frac{\partial^3\zeta_{01}^{(1)}}{\partial\xi_-\partial\eta^2}\right]$$

$$= \delta_1|\zeta_{01}^{(1)}|^2\zeta_{01}^{(1)} + \delta_2|\zeta_{01}^{(1)}|^2\zeta_{01}^{(1)} + \epsilon\left[\delta_1\zeta_{10}^{(1)^2}\left\langle\zeta_{10}^{(2)^*}\right\rangle_+ + 2\delta_1|\zeta_{10}^{(1)2}|\left\langle\zeta_{10}^{(2)}\right\rangle_+\right.$$
$$+ \delta_2\zeta_{10}^{(1)}\left\langle\zeta_{01}^{(1)}\zeta_{01}^{(2)^*}\right\rangle_+ + \delta_2\zeta_{01}^{(1)}\left\langle\zeta_{01}^{(1)^*}\zeta_{01}^{(2)}\right\rangle_+ + \delta_2\zeta_{01}^{(2)}|\zeta_{10}^{(1)}|^2 + i\delta_3|\zeta_{01}^{(1)}|^2\frac{\partial\zeta_{01}^{(1)}}{\partial\xi_-}$$
$$\left. + i\delta_4\zeta_{01}^{(1)^2}\frac{\partial\zeta_{01}^{(1)*}}{\partial\xi_-} + i\delta_7\left\langle|\zeta_{01}^{(1)}|^2\right\rangle_+\frac{\partial\zeta_{01}^{(1)}}{\partial\xi_-} + \delta_9\zeta_{01}^{(1)} + 2\zeta_{01}^{(1)}H\frac{\partial}{\partial\xi_-}\left(|\zeta_{01}^{(1)}|^2\right)\right], \tag{40}$$

where

$$\delta_9 = i\delta_5\left\langle\zeta_{10}^{(1)}\frac{\partial\zeta_{10}^{(1)^*}}{\partial\xi_+}\right\rangle_+ + i\delta_6\left\langle\zeta_{10}^{(1)^*}\frac{\partial\zeta_{10}^{(1)}}{\partial\xi_+}\right\rangle_+ - 2\left\langle H\frac{\partial}{\partial\xi_+}\left(|\zeta_{10}^{(1)}|^2\right)\right\rangle_+ .$$

In equations (39) and (40), if we put $\epsilon = 0$, $\nu = 0$ and $\gamma = 0$, then we get nonlocal mean field evolution equations in the third order (lowest order) for infinite depth water. These reduced equations become the same as equations (1b) of Pierce and Knobloch [18] for $\mu = 0$ and we proceed to the limit as $h \to \infty$.

4 Stability analysis

Equations (39) and (40) admit the solution

$$\zeta_{10} = \zeta_{10}^{(0)} = \alpha_0\ \exp(i\Delta\omega\tau), \quad \zeta_{01} = \zeta_{01}^{(0)} = \alpha_0\ \exp(i\Delta\omega\tau), \tag{41}$$

where α_0 is a real constant and the nonlinear frequency shift

$$\Delta\omega = -(\delta_1 + \delta_2)\,\alpha_0^2 . \tag{42}$$

To study modulational stability of these wave trains we introduce the perturbations

$$\zeta_{10} = \zeta_{10}^{(1)} + \epsilon\zeta_{10}^{(2)} = \zeta_{10}^{(0)}\,(1 + R_{10} + \epsilon\,, S_{10}) \tag{43}$$

$$\zeta_{01} = \zeta_{01}^{(1)} + \epsilon\zeta_{01}^{(2)} = \zeta_{01}^{(0)}\left(1 + R_{01} + \epsilon\,, S_{01}\right), \tag{44}$$

where $R_{10} = R_{10}\left(\xi_+, \eta, \tau_1, \tau_2\right)$, $R_{01} = R_{01}\left(\xi_-, \eta, \tau_1, \tau_2\right)$,
$S_{10} = S_{10}\left(\xi_+, \xi_-, \eta, \tau_1, \tau_2\right)$, $S_{01} = S_{01}\left(\xi_+, \xi_-, \eta, \tau_1, \tau_2\right)$.

We insert (43) and (44) into (39) and (40) respectively, linearize, and finally separate into real and imaginary parts to obtain the following equations in the lowest order

$$-\frac{\partial R_{10}^{\,i}}{\partial \tau_1} + \gamma_1 \frac{\partial^2 R_{10}^{\,r}}{\partial \xi_+^2} + \gamma_2 \frac{\partial^2 R_{10}^{\,r}}{\partial \eta^2} = 2\delta_1 \alpha_0^2 R_{10}^{r} + 2\delta_2 \alpha_0^2 R_{01}^{\,r}\,, \tag{45}$$

$$\frac{\partial R_{10}^{\,r}}{\partial \tau_1} + \gamma_1 \frac{\partial^2 R_{10}^{\,i}}{\partial \xi_+^2} + \gamma_2 \frac{\partial^2 R_{10}^{\,i}}{\partial \eta^2} = 0\,, \tag{46}$$

and in the next higher order,

$$\begin{aligned}
&\frac{\partial \langle S_{10}^{\,i}\rangle_-}{\partial \tau_1} - \frac{\partial R_{10}^{\,i}}{\partial \tau_2} + \gamma_1 \frac{\partial^2 \langle S_{10}^{\,r}\rangle_-}{\partial \xi_+^2} + \gamma_2 \frac{\partial^2 \langle S_{10}^{\,r}\rangle_-}{\partial \eta^2} - \gamma_4 \frac{\partial^3 R_{10}^{\,i}}{\partial \xi_+^3} - \gamma_5 \frac{\partial^3 R_{10}^{\,i}}{\partial \xi_+ \partial \eta^2} \\
&\quad = 2\delta_1 \alpha_0^2 \langle S_{10}^{r}\rangle_- + 2\delta_2 \alpha_0^2 \langle S_{01}^{\,r}\rangle_- - \delta_3 \alpha_0^2 \frac{\partial R_{10}^{\,i}}{\partial \xi_+} + \delta_4 \alpha_0^2 \frac{\partial R_{10}^{\,i}}{\partial \xi_+} + \delta_5 \alpha_0^2 \left\langle \frac{\partial R_{01}^{\,i}}{\partial \xi_-} \right\rangle_- \\
&\quad - \delta_6 \alpha_0^2 \left\langle \frac{\partial R_{01}^{\,i}}{\partial \xi_-} \right\rangle_- - \delta_7 \alpha_0^2 \frac{\partial R_{10}^{\,i}}{\partial \xi_+} + 4\alpha_0^2 H \frac{\partial R_{10}^{\,i}}{\partial \xi_+} - 4\alpha_0^2 \left\langle H \frac{\partial R_{01}^{\,r}}{\partial \xi_-} \right\rangle_-\,,
\end{aligned} \tag{47}$$

$$\begin{aligned}
&\frac{\partial \left\langle S_{10}^{\,i}\right\rangle_-}{\partial \tau_1} + \frac{\partial R_{10}^{\,i}}{\partial \tau_2} + \gamma_1 \frac{\partial^2 \left\langle S_{10}^{\,r}\right\rangle_-}{\partial \xi_+^2} + \gamma_2 \frac{\partial^2 \left\langle S_{10}^{\,r}\right\rangle_-}{\partial \eta^2} + \gamma_4 \frac{\partial^3 R_{10}^{\,i}}{\partial \xi_+^3} + \gamma_5 \frac{\partial^3 R_{10}^{\,i}}{\partial \xi_+ \partial \eta^2} \\
&\quad = \delta_3 \alpha_0^2 \frac{\partial R_{10}^{\,i}}{\partial \xi_+} + \delta_4 \alpha_0^2 \frac{\partial R_{10}^{\,i}}{\partial \xi_+} + \delta_5 \alpha_0^2 \left\langle \frac{\partial R_{01}^{\,i}}{\partial \xi_-} \right\rangle_- + \delta_6 \alpha_0^2 \left\langle \frac{\partial R_{01}^{\,i}}{\partial \xi_-} \right\rangle_- + \delta_7 \alpha_0^2 \frac{\partial R_{10}^{\,i}}{\partial \xi_+}\,.
\end{aligned} \tag{48}$$

In the above four equations the superscripts r and i indicate real and imaginary parts of the associated variables. In the transverse direction we consider the following uniform perturbations

$$R_{10}^{\,r} = p_{10} + r_{10} e^{i\lambda\xi_+} + r_{10}^{*} e^{-i\lambda\xi_+}, \qquad R_{10}^{\,i} = q_{10} + s_{10} e^{i\lambda\xi_+} + s_{10}^{*} e^{-i\lambda\xi_+},$$

$$
\begin{aligned}
R_{01}^{r} &= p_{01} + r_{01}e^{i\lambda\xi_-} + r_{01}^{*}e^{-i\lambda\xi_-}, & R_{01}^{i} &= q_{01} + s_{01}e^{i\lambda\xi_-} + s_{01}^{*}e^{-i\lambda\xi_-},\\
\langle S_{10}^{r}\rangle_- &= e_{10} + g_{10}e^{i\lambda\xi_+} + g_{10}^{*}e^{-i\lambda\xi_+}, & \langle S_{10}^{i}\rangle_- &= f_{10} + h_{10}e^{i\lambda\xi_+} + h_{10}^{*}e^{-i\lambda\xi_+},\\
\langle S_{01}^{r}\rangle_- &= e_{01} + g_{01}e^{i\lambda\xi_+} + g_{01}^{*}e^{-i\lambda\xi_+}, & \langle S_{01}^{i}\rangle_- &= f_{01} + h_{01}e^{i\lambda\xi_+} + h_{01}^{*}e^{-i\lambda\xi_+},
\end{aligned}
\tag{49}
$$

where p, q, r, s, e, f, g and h are functions of τ_1 and τ_2 only.

Here we assume dependence on τ_1 and τ_2 to be of the form $\exp(-i\Omega_1\tau_1)$ and $\exp(-i\Omega_2\tau_2)$ respectively. Now introducing perturbation relations (49) in equations (45)–(48) and equating coefficient of $e^{i\lambda\xi_+}$, on both sides we get from the lowest order equations (45) and (46)

$$
\left(\gamma_1\lambda^2 + 2\delta_1\alpha_0^2\right) r_{10} - i\Omega_1 s_{10} = 0\,, \tag{50}
$$

$$
i\Omega_1 r_{10} + \gamma_1\lambda^2 s_{10} = 0\,, \tag{51}
$$

and from equations (47) and (48) we get

$$
\begin{aligned}
&\left(\gamma_1\lambda^2 + 2\delta_1\alpha_0^2\right) g_{10} + 2\delta_2\alpha_0^2 g_{01} - i\Omega_1 h_{10} - 4|\lambda|\alpha_0^2 r_{10}\\
&\quad - i\left\{\Omega_2 + \gamma_4\lambda^3 + \lambda R_0^2\left(\delta_3 - \delta_4 + \delta_7\right)\right\} s_{10} = 0\,,
\end{aligned}
\tag{52}
$$

$$
i\Omega_1 g_{10} + \gamma_1\lambda^2 h_{10} + i\left\{\Omega_2 + \gamma_4\lambda^3 + \lambda\alpha_0^2\left(\delta_3 + \delta_4 + \delta_7\right)\right\} r_{10} = 0\,. \tag{53}
$$

The nontrivial solution of (50) and (51) is

$$
\Omega_1^2 = \gamma_1\lambda^2\left(\gamma_1\lambda^2 + 2\delta_1\alpha_0^2\right). \tag{54}
$$

Using the equations (51) and (54), we obtain the following equation from (52) and (53)

$$
\left[\Omega_1\{\Omega_2 + \gamma_4\lambda^3 + \lambda\alpha_0^2\left(\delta_3 + \delta_7\right)\} + 2\gamma_1\alpha_0^2\lambda^2|\lambda|\right] r_{10} = 0
$$

Since $r_{10} \neq 0$,

$$
\Omega_1\left\{\Omega_2 + \gamma_4\lambda^3 + \lambda\alpha_0^2\left(\delta_3 + \delta_7\right)\right\} + 2\gamma_1\alpha_0^2\lambda^2|\lambda| = 0\,. \tag{55}
$$

From equations (54) and (55) we obtain the following nonlinear dispersion relation for $\Omega = \Omega_1 + \epsilon\ \Omega_2$:

$$\Omega = \{\gamma_1\lambda^2(\gamma_1\lambda^2 + 2\delta_1\alpha_0^2)\}^{1/2} - \epsilon\{\gamma_4\lambda^3 + \lambda\alpha_0^2(\delta_3 + \delta_7)\} - 2\epsilon\gamma_1\lambda^2|\lambda|\alpha_0^2/\{\gamma_1\lambda^2(\gamma_1\lambda^2 + 2\delta_1\alpha_0^2)\}^{1/2}. \quad (56)$$

From relation (56), we observe that instability occurs when $\gamma_1\delta_1 < 0$ for long wavelengths; that is, for $\lambda \to 0^+$. When the instability condition is fulfilled, the growth rate of instability

$$\Gamma = \left[-\gamma_1\lambda^2\left(\gamma_1\lambda^2 + 2\delta_1\alpha_0^2\right)\right]^{1/2} - 2\epsilon\gamma_1\alpha_0^2\lambda^2|\lambda|\left[-\gamma_1\lambda^2\left(\gamma_1\lambda^2 + 2\delta_1\alpha_0^2\right)\right]^{1/2}. \quad (57)$$

For $\lambda^2 = -\delta_1\alpha_0^2/\gamma_1$, the maximum growth rate of the instability

$$\Gamma_m = |\delta_1|\alpha_0^2 - \frac{2\epsilon\delta_1\alpha_0^3}{\sqrt{|\gamma_1\delta_1|}}. \quad (58)$$

At marginal stability $\gamma_1\lambda^2 + 2\delta_1\alpha_0^2 = 0$, and the wave number

$$\lambda = \frac{\sqrt{2}\delta_1\alpha_0}{\sqrt{|\gamma_1\delta_1|}}. \quad (59)$$

In Figures 1 and 2 the maximum growth rate of instability Γ_m, obtained from equation (58), is plotted against wave steepness α_0 for some different values of dimensionless wind velocity ν. From the graphs, in the fourth order analysis for waves with sufficiently small wave numbers the maximum growth rate of instability Γ_m first increases with the increase of wave steepness α_0, and then it decreases with the increase of wave steepness α_0, and finally vanishes at some critical value of wave steepness α_0 beyond which there is no instability; whereas in the third order analysis the maximum growth rate of instability Γ_m increases steadily with the increase of wave steepness α_0. The growth rate is found to be appreciably higher for dimensionless wind velocity approaching its critical value. Again the wave number λ at marginal stability which obtained from equation (59) is plotted in Figure 3 and Figure 4 against wave steepness α_0 for some different values of dimensionless wind velocity ν. These figures determine the stable and unstable regions in the (λ, α_0)-plane.

5 Discussion and conclusion

The third order nonlinear evolution equations have been derived by Pierce and Knobloch [18] for two counter-propagating capillary-gravity wave packets on the surface of water of finite depth.

Here, the fourth order nonlinear evolution equations (35) and (36) are derived for two counter-propagating surface gravity wave packets in deep water in the presence of wind flowing over water. This is an extension of the evolution equations derived by Pierce and Knobloch [18] for gravity waves to one order higher for an infinite depth water and in the presence of wind flowing over it. The reason for starting from a fourth order nonlinear evolution equation is motivated by the fact, as shown by Dysthe [11], that a fourth order nonlinear evolution equation is a good starting point for making stability analysis of a uniform wave train in deep water. The evolution equations derived by us have been used to investigate the stability of a uniform standing wave train under longitudinal perturbations. An instability condition is obtained and Figures 1 and 2 plot show the maximum growth rate of instability Γ_m against wave steepness α_0 for some different values of dimensionless wind velocity ν. From the graphs it is found that in the fourth order analysis for waves with sufficiently small wave numbers the maximum growth rate of instability Γ_m first increases with the increase of wave steepness α_0, and then it decreases with the increase of wave steepness α_0, and finally vanishes at some critical value of wave steepness α_0 beyond which there is no instability; whereas in the third order analysis the maximum growth rate of instability Γ_m increases steadily with the increase of wave steepness α_0. The growth rate of instability is found to be appreciably higher for dimensionless wind velocity approaching its critical value. Our results show significant deviations from the results obtained from third order nonlinear evolution equations. Figures 3 and 4 also plot the wave number λ at marginal stability against wave steepness α_0 for some different values of dimensionless wind velocity ν in which we have obtained the stable and unstable regions in the (λ, α_0)-plane.

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A Equations for ϕ_{10}, ϕ'_{10}, ζ_{10}, ϕ_{01}, ϕ'_{01} and ζ_{01}

The equations for ϕ_{10}, ϕ'_{10} and ζ_{10} are obtained from equations (12)–(15) and (22)–(24) by setting $(m, n) = (1, 0)$. Now the solutions of equations (12) and (13) satisfying the boundary conditions (14) and (15) respectively put in the form

$$\phi_{10} = \exp\ (z\Delta_{10})A_{10}\,, \tag{60}$$
$$\phi'_{10} = \exp\ (-z\Delta_{10})A'_{10}\,, \tag{61}$$

where A_{10} and A'_{10} are functions of x_1, y_1 and t_1, and the operator

$$\exp(z\Delta_{10}) = 1 + z\Delta_{10} + \frac{z^2}{2!}\Delta_{10}^2 + \cdots\,. \tag{62}$$

Substituting the solutions (60) and (61) for ϕ_{10} and ϕ'_{10} in (22) and (23) respectively for $(m, n) = (1, 0)$ and then inverting these operators on A_{10} and A'_{10} in the linear part of the equations, we obtain

$$A_{10} = \Delta_{10}^{-1}\{a_{10} - i\omega_1\zeta_{10}\}, \tag{63}$$
$$A'_{10} = \Delta_{10}^{-1}\{i(\omega_1 - \nu k_1)\zeta_{10} - b_{10}\}, \tag{64}$$

where $\omega_1 = \omega + i\epsilon\frac{\partial}{\partial t_1}$ and $k_1 = 1 - i\epsilon\frac{\partial}{\partial x_1}$.

When contributions from nonlinear terms are dropped and only linear terms in ϵ are considered, (63) and (64) give

$$A_{10} = -i\ \omega\zeta_{10} + \epsilon\omega\frac{\partial\zeta_{10}}{\partial x_1} + \epsilon\frac{\partial\zeta_{10}}{\partial t_1}\,, \tag{65}$$

$$A'_{10} = i\,(\omega - \nu)\zeta_{10} - \epsilon\omega\frac{\partial\zeta_{10}}{\partial x_1} - \epsilon\frac{\partial\zeta_{10}}{\partial t_1}\,. \tag{66}$$

Finally substituting (60), (61) in (24) for $(m, n) = (1, 0)$ and then eliminating A_{10} and A'_{10} from the linear part by the use of (65) and (66) we get the following equation for ζ_{10}

$$\left[\omega_1^2 + \gamma(\omega_1 - \nu k_1)^2 - (1-\gamma)\Delta_{10}\right]\zeta_{10} = -i\omega_1 a_{10} - i\gamma(\omega_1 - \nu k_1) b_{10} - \Delta_{10} c_{10}\,. \tag{67}$$

Proceeding in the same way we get the following equations of ϕ_{01}, ϕ'_{01} and ζ_{01} for $(m, n) = (1, 0)$

$$\phi_{01} = \exp[z\Delta_{01}]A_{01}\,, \tag{68}$$

$$\phi'_{01} = \exp[-z\Delta_{01}]A'_{01}\,, \tag{69}$$

$$\left[\omega_1^2 + \gamma(\omega_1 - \nu k_1)^2 - (1-\gamma)\Delta_{01}\right]\zeta_{01} = -i\omega_1 a_{01} - i\gamma(\omega_1 - \nu k_1) b_{01} - \Delta_{01} c_{01}\,. \tag{70}$$

B Coefficients of the evolution equations (35) and (36)

$$\gamma_0 = \frac{2\gamma\nu\omega - 2\gamma\nu^2 + (1-\gamma)}{(1+\gamma)\,\omega^2 - \gamma\nu\omega}\,,$$

$$\gamma_1 = \frac{2\gamma\nu c_g - (1+\gamma)\,c_g^2 - \gamma\nu^2}{2\,(1+\gamma)\,\omega^2 - 2\gamma\nu\omega}\,,$$

$$\gamma_2 = \frac{(1-\gamma) - 2\gamma\nu c_g}{4\,(1+\gamma)\,\omega^2 - 4\gamma\nu\omega}\,,$$

$$\gamma_3 = \frac{2\omega c_g - (1+\gamma)\,c_g^2 + \gamma\left(\omega^2 - \nu\omega\right)}{(1+\gamma)\,\omega^2 - \gamma\nu\omega}\,,$$

$$\gamma_4 = \frac{\omega^2\left(4c_g^2 - 1\right) + (1-\gamma) - 2\gamma\nu\omega}{4\omega^2\left[2\omega^2\,(1+\gamma) - 2\omega\nu\omega\right]^2}\,,$$

$$\gamma_5 = \frac{\omega^2\left(4c_g^2 - 1\right) + 9\left(1-\gamma\right) + \gamma\omega\nu c_g}{6\left[2\omega^2\left(1+\gamma\right) - 2\gamma\nu\omega\right]^2},$$

$$\delta_1 = \left[\left(2\omega^4 + 6\omega^2\right) + \gamma\left\{\tfrac{21}{2}\left(\omega^2+\nu^2\right) + 2\left(2+p_1\right)\left(\omega-\nu\right)\left(\omega-\nu-2\omega^2\right)\right.\right.$$
$$\left.\left. - \left(1+2p_1\right)\omega + 15\omega\nu\right\} + \gamma\nu\left(\omega-\nu\right)\left(6p_1+9\right)\right]$$
$$/\left[12\omega^2 - 8\omega^4 - \gamma\left(\omega-\nu\right)^2\right],$$

$$\delta_2 = \frac{31\omega^4 - 23\omega^2 + s^2\left(1-\gamma\right) + 8\gamma\left(\omega-\nu\right)^2 - \gamma\nu\left(8-\omega p_2\right)}{8\omega^4 - 6\omega^2 - 2\gamma\left(\omega-\nu\right)^2},$$

$$\delta_3 = \frac{4\omega^2 c_g^2 - 24\left(1-\gamma\right) + 2\gamma\nu\left(\omega+\nu+p_1\omega\right)}{\omega^2\left\{\omega^2\left(1+\gamma\right) + 2\gamma\nu^2 - 3\left(1-\gamma\right)\right\}^2},$$

$$\delta_4 = \frac{12\left(1-\gamma\right) - \left(4\omega^2 c_g^2 - \omega^2 + 8\right) - \gamma\nu\left(\omega+\nu\right)}{2\omega^2\left\{\omega^2\left(1+\gamma\right) + 2\gamma\nu^2 - 3\left(1-\gamma\right)\right\}^2},$$

$$\delta_5 = \frac{\left(4\omega^2 c_g^2 + 1\right) - \left(1-\gamma\right) + 3\gamma\nu\left(\omega-\nu-p_1\nu\omega\right)}{2\omega^2\left\{\omega^2\left(1+\gamma\right) + 2\gamma\nu^2 - 3\left(1-\gamma\right)\right\}},$$

$$\delta_6 = \frac{\left(4\omega^2 c_g^2 - 1\right) + 10\left(1+\gamma\right) + \gamma\left(p_1\omega+\nu\right) + \gamma\nu\omega\left(\omega-\nu\right)}{2\omega^2\left\{4\omega^2\left(1+\gamma\right) + 2\gamma\nu^2 - 3\left(1-\gamma\right)\right\}^2},$$

$$\delta_7 = \frac{94\left(4\omega^2 c_g^2 - \omega^2\right) + 24\left(1+\gamma\right) + \gamma\nu\left(p_1\omega-\nu\right)\left(\omega-\nu\right)}{4\omega^2\left\{4\omega^2\left(1+\gamma\right) + 2\gamma\nu^2 - 3\left(1-\gamma\right)\right\}^2},$$

$$c_g = \frac{2\gamma\nu\omega - 2\gamma\nu^2 + \left(1-\gamma\right)}{2\left(1+\gamma\right)\omega - 2\gamma\nu},$$

$$p_1 = \frac{\omega^2 - 3\gamma\left(\omega-\nu\right)^2}{2\omega^2\left(\gamma-1\right) - 2\gamma\omega\left(\nu+\omega\right) + 3\left(1-\gamma\right)},$$

$$p_2 = \frac{2\omega^2\left(1+\gamma\right) - 4\gamma\nu^2}{4\omega^2\left(1+\gamma\right) + 2\gamma\nu\left(\omega-\nu\right) - 3\left(1-\gamma\right)}.$$

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