Numerical Treatment for the Fractional Fokker–Planck Equation

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Abstract

We consider a space-time fractional Fokker–Planck equation on a finite domain. The space-time fractional Fokker–Plank equation is obtained from the general Fokker–Planck equation by replacing the first order time derivative by the Caputo fractional derivative, the second order space derivative by the left and right Riemann–Liouville fractional derivatives. We propose a computationally effective implicit numerical method to solve this equation. Stability and convergence of the numerical method are discussed. We prove that the implicit numerical method is unconditionally stable, and convergent. The error estimate is also given. Numerical result is in good agreement with theoretical analysis.

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Contents

1 Introduction

There is growing interest in the field of fractional calculus. Oldham and Spanier [11], Miller and Ross [10], Samko [14] and Podlubny [12] provide the history and a comprehensive treatment of this subject. Many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by models using the theory of derivatives and integrals of fractional order. Differential equations with fractional order have recently proved to be valuable tools for the modelling of many physical phenomena [12]. A Fokker–Plank equation (FPE) has commonly been used to describe the Brownian motion of a particles [13]. An FPE describes the change of probability of a random function in space and time. The general FPE for the motion of a concentration field $u(x,t)$ has the form

$$\frac{\partial u}{\partial t} = -\nu(x)\frac{\partial u}{\partial x} + D(x)\frac{\partial^2 u}{\partial x^2}\,. \tag{1}$$

Owing to its appearance in a wide diversity of complex systems, the phenomenon of anomalous diffusion has received considerable attention in the past two decades. In particular, since the general FPE cannot describe the anomalous diffusion, several elaborate theoretical frameworks have been proposed in recent years in order to overcome this difficulty. A straightforward extension of the continuous time random walk (CTRW) model leads to a fractional Fokker–Planck Equation (FFPE). It has been demonstrated that the FPE can be generalized into a fractional FPE. FFPE was introduced with the help of a phenomenological and interesting transformation of the classical Fick law into a fractional Fick law. Metzler et al. [9] generalize the FPE and introduce the following time FFPE:

$$\frac{\partial u}{\partial t} = {}_0D_t^{1-\gamma}\left[-\nu(x)\frac{\partial u}{\partial x} + D(x)\frac{\partial^2 u}{\partial x^2}\right], \tag{2}$$

where ${}_0D_t^{1-\gamma}v(x,t)$ is the Riemann–Liouville fractional partial derivative of order $1-\gamma$.

One cannot expect to obtain a unique expression for the FFPE, since there is not a unique generalization of the differentiation to a fractional order [15].

We consider the following space-time FFPE (STFFPE):

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\nu(x,t)\frac{\partial u}{\partial x} + c_+(x,t)D_{a+}^{\gamma}u + c_-(x,t)D_{b-}^{\gamma}u + f(x,t), \tag{3}$$

where $0 < t \le T$, $a < x < b$, and the initial and boundary conditions

$$u(x,0) = \psi(x), \tag{4}$$

$$u(a,t) = \varphi_1 = 0, \quad u(b,t) = \varphi_2 = 0, \tag{5}$$

where u is solute concentration, $\nu(x,t) > 0$ and $c_+(x,t) > 0$, $c_-(x,t) > 0$ represent the average fluid velocity and the dispersion coefficient, and assume that this STFFPE has a unique and sufficiently smooth solution under the above initial and boundary conditions (some results on existence and uniqueness are developed by Ervin and Roop [2]). The time fractional derivative

$\partial^\alpha u/\partial t^\alpha$ is the Caputo fractional derivative of order α $(0 < \alpha \le 1)$ defined by [12]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\eta)}{\partial \eta} \frac{d\eta}{(t-\eta)^\alpha}\,, & 0 < \alpha < 1\,, \\ \frac{\partial u(x,t)}{\partial t}\,, & \alpha = 1\,, \end{cases} \tag{6}$$

while the space fractional derivatives $D^\gamma_{a+}u(x,t)$ and $D^\gamma_{b-}u(x,t)$ are the left and right Riemman–Liouville fractional derivatives of order γ $(1 < \gamma \le 2)$ respectively, defined by [12]

$$D^\gamma_{a+}u(x,t) = \begin{cases} \frac{1}{\Gamma(2-\gamma)} \frac{\partial^2}{\partial x^2} \int_a^x \frac{u(\xi,t)\,d\xi}{(x-\xi)^{\gamma-1}}\,, & 1 < \gamma < 2\,, \\ \frac{\partial^2 u(x,t)}{\partial x^2}\,, & \gamma = 2\,, \end{cases} \tag{7}$$

$$D^\gamma_{b-}u(x,t) = \begin{cases} \frac{1}{\Gamma(2-\gamma)} \frac{\partial^2}{\partial x^2} \int_x^b \frac{u(\xi,t)\,d\xi}{(\xi-x)^{\gamma-1}}\,, & 1 < \gamma < 2\,, \\ \frac{\partial^2 u(x,t)}{\partial x^2}\,, & \gamma = 2\,. \end{cases} \tag{8}$$

Physical considerations restrict $0 < \alpha \le 1$, $1 < \gamma \le 2$, and with the additional restriction that $u(a,t) = u(b,t) = 0$. In physical applications, this means that no tracer leaks past the left and right boundaries. The function $f(x,t)$ is used to represent sources and sinks. In the case of $\alpha = 1$ and $\gamma = 2$, the above equation reduces to the classical Fokker–Planck equation or advection-dispersion equation (ADE). When $c_+ = c_- = -0.5\cos(\pi\gamma/2)$, the STFFPE is written as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\nu(x,t)\frac{\partial u}{\partial x} - (-\Delta)^{\gamma/2}u + f(x,t), \tag{9}$$

where $-(-\Delta)^{\gamma/2} = -(-\partial^2/\partial x^2)^{\gamma/2}$ is the Riesz fractional derivative, which is a symmetric fractional generalization of the second order derivative.

The FFPE has been recently treated by a lot of people. It is presented as a useful approach for the description of transport dynamics in complex systems which are governed by anomalous diffusion and non-exponential relaxation patterns [9]. Benson et al. [1] considered space FFPE, where $c_+ = (1/2+\beta/2)$ and $c_+ = (1/2-\beta/2)$, $(-1 \le \beta \le 1)$. They gave analytic solution in terms of the α-stable error function. Liu et al. [4] considered the time FFPE and the solution was obtained by using variable transformation, Mellin and Laplace transforms, and H-functions. Huang and Liu [3] also considered the space-time FFPE and the fundamental solution was obtained by applying the Fourier–Laplace transforms. Meerschaert et al. [8] developed finite difference approximations for the space FFPE. Liu et al. [5, 6] transformed the space FFPE into a system of ordinary differential equations (Method of Lines), which was then solved using backward differentiation formulas. Yu et al. [16] developed a reliable algorithm of the a domain decomposition method to solve the linear and nonlinear space-time fractional reaction-diffusion equations in the form of a rapidly convergent series with easily computable components. They did not give its theoretical analysis. Liu et al. [7] proposed an approximation of the Lévy–Feller advection-dispersion process by random walk and finite difference method. However, numerical methods and analysis of stability and convergence for fractional partial differential equation are quite limited and difficult, and published papers on the numerical solution of the STFFPE are sparse. This motivates us to consider effective numerical methods for the STFFPE.

Section 2 proposes an implicit numerical method (INM) for the STFFPE. The stability and convergence of the STFFPE are discussed in Sections 3 and 4, respectively. Finally, some numerical examples are given in Section 5. Theoretical results are in excellent agreement with numerical testing.

2 An implicit numerical method

Define $t_k = k\tau$, $k = 0, 1, 2, \ldots, n$, $x_i = a + ih$, $i = 0, 1, 2, \ldots, m$, where $\tau = T/n$ and $h = (b - a)/m$ are time and space steps, respectively.

Let u_i^k be the numerical approximation to $u(x_i, t_k)$. Similarly, define $\nu_i^k = \nu(x_i, t_k)$, $c_{+,i}^k = c_+(x_i, t_k)$, $c_{-,i}^k = c_-(x_i, t_k)$ and $f_i^k = f(x_i, t_k)$.

Setting $\Delta_t u(x_i,t_k) = u(x_i,t_{k+1}) - u(x_i,t_k)$, then

$$\begin{aligned}\frac{\partial^\alpha u(x_i,t_{k+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\int_{t_{k-j}}^{t_{k-j+1}}\frac{u'_\eta(x_i,\eta)}{(t_{k+1}-\eta)^\alpha}\,d\eta \\ &= \frac{\tau^{-1}}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\Delta_t u(x_i,t_{k-j})\int_{t_{k-j}}^{t_{k-j+1}}\frac{d\eta}{(t_{k+1}-\eta)^\alpha}+O(\tau) \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}b_{\alpha,j}\triangle_t u(x_i,t_{k-j})+O(\tau) \qquad (10)\end{aligned}$$

where $b_{\alpha,j} = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 0,1,2,\ldots,n$. For space fractional derivatives $D^\gamma_{a+}u(x_i,t_{k+1})$ and $D^\gamma_{b-}u(x_i,t_{k+1})$, we adopted the shift Grünwald formula at level t_{k+1} [8]:

$$D^\gamma_{a+}u(x_i,t_{k+1}) = \frac{1}{h^\gamma}\sum_{j=0}^{i}g_{\gamma,j}u(x_{i-j+1},t_{k+1})+O(h), \qquad (11)$$

$$D^\gamma_{b-}u(x_i,t_{k+1}) = \frac{1}{h^\gamma}\sum_{j=0}^{m-i+1}g_{\gamma,j}u(x_{i+j-1},t_{k+1})+O(h), \qquad (12)$$

where $\quad g_{\gamma,0}=1\,,\quad g_{\gamma,j} = (-1)^j\dfrac{\gamma(\gamma-1)\cdots(\gamma-j+1)}{j!}\,,\quad j=1,2,\ldots\,.$

Using the upwind difference scheme for $\partial u/\partial x$, we have

$$\begin{aligned}\sum_{j=0}^{k}b_{\alpha,j}\triangle_t u(x_i,t_{k-j}) &= -\mu_i^{k+1}[u(x_i,t_{k+1})-u(x_{i-1},t_{k+1})] \\ &\quad + r^{(1)}_{i,k+1}\sum_{l=0}^{i+1}g_{\gamma,l}u(x_{i+1-l},t_{k+1}) \\ &\quad + r^{(2)}_{i,k+1}\sum_{l=0}^{m-i+1}g_{\gamma,l}u(x_{i-1+l},t_{k+1})\end{aligned}$$

$$+\tau^\alpha\Gamma(2-\alpha)f_i^{k+1}+R_i^{k+1}\,, \tag{13}$$

where $\mu_i^k=\nu_i^k\tau^\alpha\Gamma(2-\alpha)h^{-1}$, $r_{i,k}^{(1)}=c_{+,i}^k\tau^\alpha\Gamma(2-\alpha)h^{-\gamma}$, $r_{i,k}^{(2)}=c_{-,i}^k\tau^\alpha\Gamma(2-\alpha)h^{-\gamma}$ and

$$|R_i^k|\le C\tau^\alpha(\tau+h),\quad i=1,2,\ldots,m-1\,,\quad k=1,2,\ldots,n\,. \tag{14}$$

From (13), we obtain the following implicit difference scheme:

$$\begin{aligned}\sum_{j=0}^{k}b_{\alpha,j}\triangle_t u_i^{k-j} &= -\mu_i^{k+1}(u_i^{k+1}-u_{i-1}^{k+1})+r_{i,k+1}^{(1)}\sum_{l=0}^{i+1}g_{\gamma,l}u_{i+1-l}^{k+1}\\ &\quad +r_{i,k+1}^{(2)}\sum_{l=0}^{m-i+1}g_{\gamma,l}u_{i-1+l}^{k+1}+\tau^\alpha\Gamma(2-\alpha)f_i^{k+1}\,,\end{aligned} \tag{15}$$

where $0<i<m$, $0\le k<n$. Hence, we have

$$\begin{aligned}u_i^{k+1}&=b_{\alpha,k}u_i^0+\sum_{j=0}^{k-1}(b_{\alpha,j}-b_{\alpha,j+1})u_i^{k-j}-\mu_i^{k+1}(u_i^{k+1}-u_{i-1}^{k+1})\\ &\quad +r_{i,k+1}^{(1)}\sum_{l=0}^{i+1}g_{\gamma,l}u_{i+1-l}^{k+1}+r_{i,k+1}^{(2)}\sum_{l=0}^{m-i+1}g_{\gamma,l}u_{i-1+l}^{k+1}+\tau^\alpha\Gamma(2-\alpha)f_i^{k+1}\end{aligned} \tag{16}$$

with the boundary and initial conditions

$$u_i^0=\psi(ih),\quad u_0^k=0\,,\quad u_m^k=0\,, \tag{17}$$

where $k=0,1,2,\ldots,n$, $i=0,1,2,\ldots,m$.

When $\alpha=1$, $\mu_i^k=\nu_i^k\tau h^{-1}$, $r_{i,k}^{(1)}=c_{+,i}^k\tau h^{-\gamma}$, $r_{i,k}^{(2)}=c_{-,i}^k\tau h^{-\gamma}$, the equation (16) is rewritten as

$$\begin{aligned}u_i^{k+1} &= u_i^k-\mu_i^{k+1}(u_i^{k+1}-u_{i-1}^{k+1})\\ &\quad +r_{i,k+1}^{(1)}\sum_{l=0}^{i+1}g_{\gamma,l}u_{i+1-l}^{k+1}+r_{i,k+1}^{(2)}\sum_{l=0}^{m-i+1}g_{\gamma,l}u_{i-1+l}^{k+1}+\tau f_i^{k+1}\,.\end{aligned} \tag{18}$$

Lemma 1 *The coefficients* $b_{\alpha,j}$, $g_{\gamma,j}$, $j = 0, 1, 2, \ldots$, *satisfy*

1. $b_{\alpha,0} = 1$, $b_{\alpha,j} > 0$, $j = 1, 2, \ldots$;
2. $b_{\alpha,j} > b_{\alpha,j+1}$, $j = 0, 1, \ldots$;
3. $g_{\gamma,0} = 1$, $g_{\gamma,1} = -\gamma < 0$ *and* $g_{\gamma,j} > 0$, $j = 2, 3, \ldots$;
4. $g_{\gamma,0} + g_{\gamma,1} + g_{\gamma,2} + \cdots = 0$, *and for* $i = 1, 2, \ldots$, *we have* $g_{\gamma,0} + g_{\gamma,1} + \cdots + g_{\gamma,i} < 0$.

When $0 < \alpha < 1$, from

$$\lim_{k\to\infty} \frac{b_{\alpha,k}^{-1}}{k^{\alpha}} = \lim_{k\to\infty} \frac{k^{-1}}{(1+\frac{1}{k})^{1-\alpha} - 1} = \frac{1}{1-\alpha} ,$$

we obtain the following lemma.

Lemma 2 *If* $0 < \alpha < 1$, *there is a positive constant* C *such that*

$$b_{\alpha,k}^{-1} \le C k^{\alpha} , \quad k = 0, 1, 2, \ldots . \tag{19}$$

3 Stability of the implicit numerical method

This section discusses the stability of the INM. We rewrite (16) as

$$\begin{aligned} u_i^{k+1} + \mu_i^{k+1}(u_i^{k+1} - u_{i-1}^{k+1}) - r_{i,k+1}^{(1)} \sum_{l=0}^{i+1} g_{\gamma,l} u_{i+1-l}^{k+1} - r_{i,k+1}^{(2)} \sum_{l=0}^{m-i+1} g_{\gamma,l} u_{i-1+l}^{k+1} \\ = b_{\alpha,k} u_i^0 + \sum_{j=0}^{k-1} (b_{\alpha,j} - b_{\alpha,j+1}) u_i^{k-j} + \tau^{\alpha} \Gamma(2-\alpha) f_i^{k+1} . \end{aligned} \tag{20}$$

Suppose that

$$\|\mathbf{u}^{k+1}\|_\infty = \max_{1\le i\le m-1} |u_i^{k+1}|\,, \quad \|\mathbf{f}\|_\infty = \max_{0\le i\le m\,,\quad 0\le k\le n} |f_i^k|\,,$$

then we can obtain the following theorem.

Theorem 3 *Suppose that* u_i^k, $i=1,2,\ldots,m-1$, $k=1,2,\ldots,n$, *is the solution of (16), then*

$$\|\boldsymbol{u}^k\|_\infty \le \|\boldsymbol{u}^0\|_\infty + C\|\boldsymbol{f}\|_\infty\,, \quad k=1,2,\ldots,n\,. \tag{21}$$

Proof: For $0<\alpha<1$, we can obtain

$$\|\mathbf{u}^k\|_\infty \le \|\mathbf{u}^0\|_\infty + C_1 k^\alpha \tau^\alpha \|\mathbf{f}\|_\infty\,, \quad k=1,2,\ldots,n\,.$$

Assume that $|u_{i_0}^1| = \max\{|u_i^1|, |u_i^2|, \ldots, |u_i^{m-1}|\}$, using $\sum_{l=0}^{i_0+1} g_{\gamma,l} < 0$ and $\sum_{l=0}^{m-i_0+1} g_{\gamma,l} < 0$, we have

$$\begin{aligned} |u_{i_0}^1| &\le (1+\mu_{i_0}^{k+1})|u_{i_0}^1| - \mu_{i_0}^{k+1}|u_{i_0-1}^1| \\ &\quad -r_{i_0,k+1}^{(1)} \textstyle\sum_{l=0}^{i_0+1} g_{\gamma,l}|u_{i_0-l+1}^1| - r_{i_0,k+1}^{(2)} \textstyle\sum_{l=0}^{m-i_0+1} g_{\gamma,l}|u_{i_0+l-1}^1| \\ &\le |u_{i_0}^1 + \mu_{i_0}^{k+1}u_{i_0}^1 - \mu_{i_0}^{k+1}u_{i_0-1}^1 \\ &\quad -r_{i_0,k+1}^{(1)} \textstyle\sum_{l=0}^{i_0+1} g_{\gamma,l}u_{i_0-l+1}^1 - r_{i_0,k+1}^{(2)} \textstyle\sum_{l=0}^{m-i_0+1} g_{\gamma,l}u_{i_0+l-1}^1| \\ &= |u_{i_0}^0 + \tau^\alpha\Gamma(2-\alpha) f_{i_0}^1|. \end{aligned}$$

Thus, $\|\mathbf{u}^1\|_\infty \le \|\mathbf{u}^0\|_\infty + b_{\alpha,0}^{-1}\tau^\alpha\Gamma(2-\alpha)\|\mathbf{f}\|_\infty$. Suppose that $\|\mathbf{u}^j\|_\infty \le \|\mathbf{u}^0\|_\infty + b_{\alpha,j-1}^{-1}\tau^\alpha\Gamma(2-\alpha)\|\mathbf{f}\|_\infty$, $j=1,2,\ldots,k$. Using $b_{\alpha,j}^{-1} \le b_{\alpha,k}^{-1}$, $j=0,1,\ldots,k-1$, we have

$$\|\mathbf{u}^j\|_\infty \le \|\mathbf{u}^0\|_\infty + b_{\alpha,k}^{-1}\tau^\alpha\Gamma(2-\alpha)\|\mathbf{f}\|_\infty\,, \quad j=1,2,\ldots,k\,. \tag{22}$$

Similarly, let $|u_{i_0}^{k+1}| = \max\{|u_1^{k+1}|, |u_2^{k+1}|, \ldots, |u_{m-1}^{k+1}|\}$, we also have

$$\begin{aligned} |u_{i_0}^{k+1}| &\le (1+\mu_{i_0}^{k+1})|u_{i_0}^{k+1}| - r_1|u_{i_0-1}^{k+1}| \\ &\quad -r_{i_0,k+1}^{(1)} \textstyle\sum_{l=0}^{i_0+1} g_{\gamma,l}|u_{i_0-l+1}^{k+1}| - r_{i_0,k+1}^{(2)} \textstyle\sum_{l=0}^{m-i_0+1} g_{\gamma,l}|u_{i_0+l-1}^{k+1}| \\ &\le |u_{i_0}^{k+1} + \mu_{i_0}^{k+1}u_{i_0}^{k+1} - \mu_{i_0}^{k+1}u_{i_0-1}^{k+1} \\ &\quad -r_{i_0,k+1}^{(1)} \textstyle\sum_{l=0}^{i_0+1} g_{\gamma,l}u_{i_0-l+1}^{k+1} - r_{i_0,k+1}^{(2)} \textstyle\sum_{l=0}^{m-i_0+1} g_{\gamma,l}u_{i_0+l-1}^{k+1}|. \end{aligned}$$

Therefore,

$$|u_{i_0}^{k+1}| \le |b_{\alpha,k}u_{i_0}^0 + \sum_{j=0}^{k-1}(b_{\alpha,j} - b_{\alpha,j+1})u_{i_0}^{k-j} + \tau^\alpha\Gamma(2-\alpha)f_{i_0}^{k+1}|.$$

Using (22), we have

$$\begin{aligned} \|\mathbf{u}^{k+1}\|_\infty &\le b_{\alpha,k}\|\mathbf{u}^0\|_\infty + \textstyle\sum_{j=0}^{k-1}(b_{\alpha,j} - b_{\alpha,j+1})\|\mathbf{u}^{k-j}\|_\infty + \tau^\alpha\Gamma(2-\alpha)\|\mathbf{f}\|_\infty \\ &\le b_{\alpha,k}\|\mathbf{u}^0\|_\infty + \textstyle\sum_{j=0}^{k-1}(b_{\alpha,j} - b_{\alpha,j+1})\|\mathbf{u}^0\|_\infty \\ &\quad + b_{\alpha,k}^{-1}[\textstyle\sum_{j=0}^{k-1}(b_{\alpha,j} - b_{\alpha,j+1}) + b_{\alpha,k}]\tau^\alpha\Gamma(2-\alpha)\|\mathbf{f}\|_\infty \\ &\le \|\mathbf{u}^0\|_\infty + b_{\alpha,k}^{-1}\tau^\alpha\Gamma(2-\alpha)\|\mathbf{f}\|_\infty . \end{aligned} \tag{23}$$

For $\alpha = 1$, similarly, we obtain $\|\mathbf{u}^{k+1}\|_\infty \le \|\mathbf{u}^k\|_\infty + \tau\|\mathbf{f}\|_\infty$. Using mathematical induction, we have $\|\mathbf{u}^k\|_\infty \le \|\mathbf{u}^0\|_\infty + k\tau\|\mathbf{f}\|_\infty$, $k = 1, 2, \ldots, n$.

From (19) and $k\tau \le T$, the theorem is obtained. ♠

We suppose that $\tilde{u}_i^k$, $(0 \le i \le m\,,\ 0 \le j \le n)$ is the approximate solution of (16) and (17), the error $\varepsilon_i^k = \tilde{u}_i^k - u_i^k$, $(0 \le i \le m\,,\ 0 \le k \le n)$ satisfies

$$\begin{aligned} \varepsilon_i^{k+1} &= b_{\alpha,k}\varepsilon_i^0 + \sum_{j=0}^{k-1}(b_{\alpha,j} - b_{\alpha,j+1})\varepsilon_i^{k-j} - \mu_i^{k+1}(\varepsilon_i^{k+1} - \varepsilon_{i-1}^{k+1}) \\ &\quad + r_{i,k+1}^{(1)}\sum_{l=0}^{i+1} g_{\gamma,l}\varepsilon_{i+1-l}^{k+1} + r_{i,k+1}^{(2)}\sum_{l=0}^{m-i+1} g_{\gamma,l}\varepsilon_{i-1+l}^{k+1} . \end{aligned} \tag{24}$$

Applying Theorem 3, we obtain $\|\mathbf{E}^k\|_\infty \le \|\mathbf{E}^0\|_\infty$, $k = 1, 2, \ldots, n$, where $\|\mathbf{E}^k\|_\infty = \max\{|\varepsilon_1^k|, |\varepsilon_2^k|, \ldots, |\varepsilon_{m-1}^k|\}$. Thus, the following theorem is valid.

Theorem 4 *The fractional implicit difference approximations defined by (16) and (17) are unconditionally stable.*

4 Convergence of the implicit numerical method

Let $u(x_i, t_k)$, $(i = 1, 2, \ldots, m-1\,,\ k = 1, 2, \ldots, n)$ be the exact solution of the equations (3)–(5) at mesh point (x_i, t_k). Define $\eta_i^k = u(x_i, t_k) - u_i^k$, $i = 1, 2, \ldots, m-1\,,\ k = 1, 2, \ldots, n$ and $\mathbf{Y}^k = (\eta_1^k, \eta_2^k, \ldots, \eta_{m-1}^k)^T$.

Using $\mathbf{Y}^0 = 0$, substitution into (16) leads to

$$\eta_i^{k+1} + \mu_i^{k+1}(\eta_i^{k+1} - \eta_{i-1}^{k+1}) - r_{i,k+1}^{(1)} \sum_{l=0}^{i+1} g_{\gamma,l}\eta_{i+1-l}^{k+1} - r_{i,k+1}^{(2)} \sum_{l=0}^{m-i+1} g_{\gamma,l}\eta_{i-1+l}^{k+1}$$
$$= b_{\alpha,k}\eta_i^0 + \sum_{j=0}^{k-1}(b_{\alpha,j} - b_{\alpha,j+1})\eta_i^{k-j} + R_i^{k+1}\,, \tag{25}$$

where $i = 1, 2, \ldots, m-1\,,\ k = 0, 1, 2, \ldots, n-1$.

Applying Theorem 3, with (14), we obtain

$$\|\mathbf{Y}^k\|_\infty \leq C\tau^{-\alpha}\|\mathbf{R}^k\|_\infty \leq C(\tau + h).$$

Therefore, the following theorem is valid.

Theorem 5 *Let u_i^k be the numerical solution computed by use of the* INM *(16) and (17). Then there is a positive constant C, such that*

$$|u_i^k - u(x_i, t_k)| \leq C(\tau + h), \quad i = 1, 2, \ldots, m-1\,, \quad k = 1, 2, \ldots, n\,. \tag{26}$$

5 Numerical result

This section presents an example to demonstrate that the INM is a computationally effective method, and the computed result is in good agreement with theoretical analysis.

Table 1: The maximum error between the exact solution and the numerical solution $\max|u_i^k - u(x_i,t_k)|$.

Δt	Δx	$\beta=-1$	$\beta=-0.5$	$\beta=0$	$\beta=0.5$	$\beta=1.0$
0.1	0.1	8.6E-2	8.2E-2	8.3E-2	9.0E-2	1.0E-1
0.05	0.05	5.2E-2	5.4E-2	5.7E-2	6.2E-2	6.8E-2
0.025	0.025	2.9E-2	3.0E-2	3.3E-2	3.5E-2	3.8E-2
0.0125	0.0125	1.5E-2	1.6E-2	1.7E-2	1.9E-2	2.0E-2

We consider the following STFFPE

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -\frac{\partial u}{\partial x} + \left(\frac{1}{2}+\frac{\beta}{2}\right) D^\gamma_{a+}u + \left(\frac{1}{2}-\frac{\beta}{2}\right) D^\gamma_{b-}u + f(x,t),$$

where $0<t\leq 1$, $0<x<1$, and the initial and boundary conditions are

$$u(x,0) = 21x^2(1-x)^2\,, \tag{27}$$
$$u(0,t)=0\,,\quad u(1,t)=0\,, \tag{28}$$

and

$$\begin{aligned} f(x,t) &= 10\Gamma(\alpha+2)tx^2(1-x)^2 + 2(21+10t^{\alpha+1})(x-3x^2+2x^3)\\ &-(1+\beta)(21+10t^{1+\alpha})\left[\frac{x^{2-\gamma}}{\Gamma(3-\gamma)} - \frac{6x^{3-\gamma}}{\Gamma(4-\gamma)} + \frac{12x^{4-\gamma}}{\Gamma(5-\gamma)}\right]\\ &-(1-\beta)(21+10t^{1+\alpha})\left[\frac{(1-x)^{2-\gamma}}{\Gamma(3-\gamma)} - \frac{6(1-x)^{3-\gamma}}{\Gamma(4-\gamma)} + \frac{12(1-x)^{4-\gamma}}{\Gamma(5-\gamma)}\right]. \end{aligned}$$

The exact solution of the above equation is $u(x,t)=(21+10t^{1+\alpha})x^2(1-x)^2$.

Table 1 shows the maximum absolute numerical error, at time $t=1.0$, between the exact solution and the numerical solution obtained by INM. From Table 1, our INM yields convergence with $O(\tau+h)$.

6 Conclusions

An INM for the STFFPE in a bounded domain has been described and demonstrated. We prove that the INM is unconditionally stable and convergent. This method and technique can be applied to solve fractional (in space and in time) partial differential equations.

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References

[1] D. A. Benson, S. W. Wheatcraft, M.M Meerschaert, Application of a fractional advection-dispersion equation, *Water Resour. Res.*, **36**(6) (2000) 1403–1412. doi:10.1029/2000WR900031 C762

[2] J. S. Ervin, J. P. Roop, Variational solution of fractional advection dispersion equations on bounded domains in R^d, *Numer. Meth. P. D. E.*, **23**(2) (2007) 256–281. doi:10.1002/num.20169 C761

[3] F. Huang, F. Liu, The fundamental solution of the space-time fractional advection-dispersion equation, *J. Appl. Math. Computing*, **18** (2005) 233–245. http://jamc.net/contents/table_contents_view.php?idx=223 C763

[4] F. Liu, V. Anh, I. Turner and P. Zhuang, Time fractional advection dispersion equation, *J. Appl. Math. Computing*, **13** (2003) 233–245.

http://jamc.net/contents/table_contents_view.php?idx=74 C763

[5] F. Liu, V.Anh, I. Turner, Numerical solution of space fractional Fokker–Planck equation, *J. Comp. and Appl. Math.*, **166** (2004) 209–219. doi:10.1016/j.cam.2003.09.028 C763

[6] F. Liu, V. Anh, I. Turner and P. Zhuang, Numerical simulation for solute transport in fractal porous media, *ANZIAM J.*, **45**(E) (2004) 461–473. http://anziamj.austms.org.au/V45/CTAC2003/Liuf C763

[7] Q. Liu, F. Liu, I. Turner and V. Anh, Approximation of the Lévy–Feller advection-dispersion process by random walk and finite difference method, *J. Phys. Comp.*, **222** (2007) 57–70. doi:10.1016/j.jcp.2006.06.005 C763

[8] M. Meerschaert, C. Tadjeran, Finite difference approximations for fractional advection-dispersion flow equations, *J. Comp. and Appl. Math.*, **172** (2004) 65–77. doi:10.1016/j.cam.2004.01.033 C763, C764

[9] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* **339** (2000) 1–77. doi:10.1016/S0370-1573(00)00070-3 C761, C762

[10] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, New York: John Wiley, 1993. C760

[11] K. B. Oldham, J. Spanier, *The fractional calculus*, New York and London: Academic Press, 1974. C760

[12] I. Podlubny, *Fractional Differential Equations*, Academic, Press, New York, 1999. C760, C762

[13] H. Risken, *The Fokker–Planck Equations*, Springer, Berlin, 1988. C760

[14] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, USA: Gordon and Breach Science Publishers, 1993. C760

[15] V. V. Yanovsky, A. V. Chechkin, D. Schertzer and A. V. Tur, Lévy anomalous diffusion and fractional Fokker-Planck equation, *Physica A*, **282**, (2000), 13–34. doi:10.1016/S0378-4371(99)00565-8 C761

[16] Q. Yu, F. Liu, V. Anh and I. Turner, Solving linear and nonlinear space-time fractional reaction-diffusion equations by a domian decomposition method, *International J. for Numer. Meth. In Eng.*, (2007), in press. C763

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