An always convergent method for finding the spectral radius of an irreducible non-negative matrix

R. J. Wood* M. J. O'Neill†

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Abstract

An always convergent method is used to calculate the spectral radius of an irreducible non-negative matrix. The method is an adaptation of a method of Collatz (1942), and has similarities to both the power method and the inverse power method. For large matrices it is faster than the eig routine in Matlab. Special attention is paid to the step-by-step improvement of the bounds and the subsequent convergence of this method.

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*School of IT, Charles Sturt University, Australia. mailto:rwood@csu.edu.au
†Charles Sturt University

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1 Introduction

Calculating the spectral radius of a matrix is useful in a number of applications. Our particular interest has been in the area of mathematical economics known as Input-Output Analysis, where finding the spectral radius of a non-negative matrix is an important technique in verifying that a particular linear system has a unique positive solution (see Wood & O'Neill [8]). Other applications include finding the roots of a polynomial equation using a companion matrix, determining whether a Hessian matrix is positive definite, finding the first few eigenvalues of a covariance matrix (an important statistical technique in factor analysis), calculating the 2-norm of a matrix (that is, $\|A\|_2 = \sqrt{\rho(A^T A)}$), verifying that a matrix is convergent ($\lim_{n\to\infty} A^n = 0$, which occurs if and only if $\rho(A) < 1$). It is acknowledged that some of the applications listed above do not always involve non-negative matrices but, when they do, the technique of this paper is appropriate. In this paper, consideration is given to several methods for calculating the spectral radius, and the advantages and disadvantages of each are considered.

2 The power method

This is a well-known method for approximating the dominant eigenvalue, λ_1, of a matrix. In this paper we restrict our discussion to that of a non-

negative matrix A. The method proceeds by choosing an initial vector q_0 and performing the iterations $q_\nu = Aq_{\nu-1} = \cdots = A^\nu q_0$, $\nu \geq 1$. Typically, q_ν tends to the dominant eigenvector, and the dominant eigenvalue is usually obtained by one of two methods:

1. calculation of $\lambda_1^{(\nu)}$ where $\lambda_1^{(\nu)} = u^T A q_\nu / u^T q_\nu$, and u is chosen so that $u^T q_0 \neq 0$ (see Conte and de Boor [3, p.192]). A smoother variant of this replaces u by q_ν (see Golub & Van Loan [4, p.326]).

2. comparison of two corresponding non-zero components of $q_{\nu+1}$ and q_ν (see Atkinson [1, p.604]). The ratio of these components tends to λ_1.

In order to avoid overflow and underflow appropriate scaling of q_ν is carried out at each step. The convergence ratio of the method is typically $|\lambda_2/\lambda_1|$ where λ_1 is the dominant eigenvalue and λ_2 is a subdominant eigenvalue. Difficulties can occur with the method if the matrix A has two eigenvalues of maximum magnitude, or if λ_1 and λ_2 have approximately equal magnitudes. Difficulties can also occur if q_0 does not have a component in the direction of x_1, the dominant eigenvector. Convergence will then be to a subdominant eigenvalue. It is worth noting in passing that rounding errors may intervene and produce eventual convergence to λ_1, even in this case (see Stewart [6, p.343]). However, in the case of a non-negative matrix, a strictly positive initial vector will always have a positive component in the direction of x_1. This is proved in the following theorem, which applies to any non-negative matrix, reducible or irreducible.

Theorem 1 *Let matrix $A \geq 0$, with Jordan basis $x_1, x_2, \ldots, x_n$, corresponding to the eigenvalues $\lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_k| > |\lambda_{k+1}| > \cdots > |\lambda_n|$ have an initial $m \times m$ $(1 \leq m \leq k)$ Jordan block associated with λ_1. Let q_0 be an arbitrary positive vector which when written in terms of this basis is $q_0 = \gamma_1 x_1 + \gamma_2 x_2 + \cdots + \gamma_n x_n$. Then it is guaranteed that $\gamma_m > 0$.*

Proof: A can be factorised in the form

$$A = X \begin{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_1 \end{bmatrix} & 0 \\ 0 & J \end{bmatrix} X^{-1}, \quad \text{where} \quad X = [x_1, x_2, \ldots, x_n] \qquad (1)$$

(See Halmos [5, p.112ff] for verification that the factorisation (1) is possible.)

If we pre-multiply both sides of (1) by $X^{-1} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$,

$$\begin{bmatrix} y_1^T \\ \vdots \\ y_m^T \\ \vdots \\ y_n^T \end{bmatrix} A = \begin{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_1 \end{bmatrix} & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} y_1^T \\ \vdots \\ y_m^T \\ \vdots \\ y_n^T \end{bmatrix}$$

Therefore $y_m^T A = \lambda_1 y_m^T$, which means y_m^T is a left eigenvector of A corresponding to the dominant eigenvalue, λ_1. ($y_m^T \neq 0$, since X^{-1} cannot have a complete row of zeros.) Therefore $y_m^T \geq 0$, by the well-known generalisation of the Perron-Frobenius theorem for any non-negative matrix (see Varga [7, Theorem 2.7, p.46]).

Now, $y_m^T q_0 = y_m^T(\gamma_1 x_1 + \cdots + \gamma_m x_m + \cdots + \gamma_n x_n) = \gamma_m > 0$, since $y_m^T q_0 > 0$. $y_m^T q_0 = \gamma_m$, since if x is a right eigenvector of A corresponding to λ_1 and y is a left eigenvector corresponding to $\lambda_m \neq \lambda_1$, then $y^T x = 0$ (see Stewart [6, p.272]).

♠

The theorem can obviously be extended to any $m \times m$ Jordan block for λ_1. We now show how the Power Method performs on a selection of test matrices, our aim in including these examples is to show the deficiencies of some rival methods. In each case, q_0 was chosen as $[1, 1, \ldots, 1]$.

Example 2

$$A1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 4 & 5 \end{bmatrix},$$

$\lambda_1 = 5.7287$, $\lambda_2 = -0.3644 + 0.2045i$, $\lambda_3 = -0.3644 - 0.2045i$. This matrix has a single dominant eigenvalue. Using Method 2 described above the method converges in 6 iterations to the correct 4 decimal value. The iterates are 10, 5.5, 5.7455, 5.7278, 5.7287 and 5.7287.

Example 3

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

This matrix is cyclic, it has one real and two complex eigenvalues all of which are equal in magnitude. In particular $|\lambda_1| = |\lambda_2| = |\lambda_3| = \sqrt[3]{6}$. Using Method 1, it does not converge, it cycles among the values 2, 1.8333, and 1.6364.

Example 4

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$\lambda_1 = 1.2720$, $\lambda_2 = -1.2720$, $\lambda_3 = 0.7862i$, $\lambda_4 = -0.7862i$. This matrix is also cyclic and has two dominant eigenvalues which are real, and of equal magnitude but opposite sign. Using Method 1, it does not converge, it cycles between the values 1.2361 and 1.3090.

Example 5

$$D = \begin{bmatrix} 0.92 & 0.0001 \\ 0.0002 & 0.91 \end{bmatrix}$$

$\lambda_1 = 0.9200$, $\lambda_2 = 0.9100$. This matrix has two eigenvalues, which are both real and approximately equal in magnitude. When Method 1 is applied

to the above matrix, the method converges to the incorrect value of 0.9121 after two iterations. This occurs if an absolute value convergence criterion is applied to the eigenvalue approximations. However, if a different convergence criterion such as suggested in Golub & Van Loan [4, p.332] is applied then such premature convergence is avoided, but convergence is very slow with only one decimal place of accuracy after 100 iterations. The Golub & Van Loan procedure uses an approximation to the left and right eigenvectors to estimate the error at each step. The advantage of the Golub & Van Loan procedure is that it avoids premature convergence to the wrong value.

Using Method 2 with the Golub & Van Loan convergence criterion, again false premature convergence is avoided, but convergence is also very slow, with only two decimal places of accuracy after 100 iterations.

Example 6

$$E = \begin{bmatrix} 0.92 & 1 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0.92 \end{bmatrix}$$

$\lambda_1 = 0.92\,,\ \lambda_2 = 0.92\,,\ \lambda_3 = 0.5\,.$ This matrix has two dominant eigenvalues which are real and equal. Furthermore, λ_1 and λ_2 occur in a 2×2 Jordan Block. Admittedly, this matrix is reducible but can be made irreducible by adding 10^{-6} to the (3,1) element. The effect of this perturbation can be estimated by using the result from Stewart [6] quoted later in the Conclusion. When Method 2 was applied with the Golub and Van Loan convergence criterion, convergence was extremely slow with only one decimal place of accuracy after 100 iterations. The purpose of the inclusion of this example is to show that, even if a non-negative matrix is reducible, modifications can be made to estimate its spectral radius. However, great care should be taken in perturbing defective matrices. It is advisable to estimate the relevant quantities in the above-mentioned result from Stewart [6]. This will give a measure of the appropriateness of the perturbation.

3 An alternative method

In order to overcome some of the above difficulties, a more robust method is proposed. This method is based upon a result by Collatz [2].

Theorem 7 *Let $A \geq 0$ be an $n \times n$ irreducible matrix and q_0 be an arbitrary positive n-dimensional vector. Defining $q_\nu = Aq_{\nu-1} = \cdots = A^\nu q_0$, $\nu \geq 1$, let*

$$\underline{\lambda}_\nu = \min_{1 \leq i \leq n} \frac{q_{\nu+1}^{(i)}}{q_\nu^{(i)}} \quad \textit{and} \quad \bar{\lambda}_\nu = \max_{1 \leq i \leq n} \frac{q_{\nu+1}^{(i)}}{q_\nu^{(i)}},$$

where the superscript i represents the ith component of a vector. Then, denoting the spectral radius of A by $\rho(A)$,

$$\underline{\lambda}_0 \leq \underline{\lambda}_1 \leq \underline{\lambda}_2 \leq \cdots \leq \rho(A) \leq \cdots \leq \bar{\lambda}_2 \leq \bar{\lambda}_1 \leq \bar{\lambda}_0 .$$

For the matrix $A1$ in Example 2, convergence is quite rapid. The bounds on $\underline{\lambda}_\nu$ and $\bar{\lambda}_\nu$, are (1,10), (1,10), (5.5,10), (5.5,5.7455), (5.7278,5.7455) and (5.7278,5.7287). However, for the matrix B in Example 3, the bounds do not improve from the initial bounds produced of $(1, 3)$. An explanation of the different behaviour of matrices $A1$ and B is provided in the following result mentioned in Varga [7].

Theorem 8 *In Theorem 7 both the sequences $\{\underline{\lambda}_\nu\}_{\nu=0}^{\infty}$ and $\{\bar{\lambda}_\nu\}_{\nu=0}^{\infty}$ converge to $\rho(A)$, from an arbitrary initial positive vector q_0, if and only if the irreducible matrix $A \geq 0$ is primitive.*

Noting that $A1$ is primitive and B is cyclic, we then have an explanation of their different behaviours. The method of Theorem 7 is obviously closely related to the Power Method, but it has the double advantage of being always convergent when A is primitive, and also, of providing an estimate of the error at each step and this overcomes the problem of premature convergence to the wrong value. It is worth noting that a cyclic matrix can always be converted

to a primitive matrix by means of a positive spectral shift. This is proved in the next theorem.

Theorem 9 *If the matrix $A \geq 0$ is an $n \times n$ irreducible matrix then the matrix $qI + A$, where $q > 0$, is primitive.*

Proof: If $A \geq 0$ and irreducible then it has a unique, positive eigenvalue equal to its spectral radius $\rho = \rho(A)$ by the Perron-Frobenius Theorem. If the matrix qI is added to the matrix A, then this will produce another unique, positive eigenvalue equal to the spectral radius $\rho + q$ of the matrix $A + qI$. Further there will be no other eigenvalues of $A + qI$ with modulus equal to $\rho + q$. Hence the matrix $A + qI$ is primitive. ♠

Theorem 9 can be used to ensure convergence for Matrix B, by applying the method of Theorem 7 to the matrix $B1 = B + I$, where I is the 3×3 identity matrix. $B1$ is then a primitive matrix and after 15 iterations, the approximation $\rho(B1) = 2.8171$ is obtained.

Hence $\rho(B) = \rho(B1) - 1 = 1.8171$. The question then arises as to what might be an optimal shift to give the most favourable rate of convergence. This is not easily answered except in the case of a real symmetric matrix. As remarked by Stewart [6, p.342] the search for an optimal shift is not very satisfactory in automatic computation. An alternative approach is adopted in this paper. We apply the method of Theorem 7 to the matrix $(qI - A)^{-1}$ instead of the matrix A. But first it is necessary to show that $(qI - A)^{-1}$ is primitive.

Theorem 10 *If $A \geq 0$ is an $n \times n$ irreducible matrix with $\rho(A) < q$, then $(qI - A)^{-1}$ is a non-negative irreducible matrix. Furthermore it is primitive.*

Proof: We first show $(qI - A)^{-1}$ is non-negative. Since $\rho(A) < q$,

$$(qI - A)^{-1} = \frac{1}{q}\left(I + \frac{A}{q} + \left(\frac{A}{q}\right)^2 + \cdots + \left(\frac{A}{q}\right)^n + \cdots\right). \tag{2}$$

Therefore, since A is non-negative, $(qI - A)^{-1}$ is also non-negative. Furthermore, the series (2) must be irreducible since A is irreducible. We next show the matrix $(qI - A)^{-1}$ is primitive.

Since $A \geq 0$ and irreducible with $q > \rho(A)$, the unique eigenvalue closest to zero of $(qI - A)$ is $(q - \rho(A))$; so $1/(q - \rho(A))$ is the unique dominant eigenvalue of $(qI - A)^{-1}$ and hence its spectral radius. This proves that $(qI - A)^{-1}$ is primitive. ♠

Unfortunately, if A is reducible, $(qI - A)^{-1}$ may also be reducible. Example 11 shows such a case.

Example 11 Both A and $(qI - A)^{-1}$ are reducible when $q = 3$ and

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \text{whence} \quad (qI - A)^{-1} = \begin{bmatrix} 0.5 & 0.25 \\ 0 & 0.5 \end{bmatrix}.$$

Corollary 12 *If A is an irreducible, non-negative matrix, and $\rho(A) < q$, then the method of Theorem 7 applied to the matrix $(qI - A)^{-1}$ is certain to converge.*

Proof: Follows from Theorems 7, 8 and 10. ♠

In this method it is not necessary to explicitly calculate $(qI - A)^{-1}$, but merely calculate the solution of the linear system $(qI - A)x = y$ at each iteration, noting that the once only LU decomposition will suffice for each iteration.

This method is obviously closely related to the Inverse Power Method (see Stewart [6, pp.343–5]), but has several advantages: it is always convergent

when A is irreducible and it gives an estimate of the error at each step. It also has an advantage over the method of Theorem 7 in that it applies to any irreducible, non-negative matrix, not just a primitive matrix. Since $\rho(A) \leq \|A\|_\infty$, choosing $q > \|A\|_\infty$ will ensure $\rho(A) < q$. This method was applied to Examples 2–6, with the following results:

- Example 2 converged in 17 iterations;
- Example 3 converged in 8 iterations;
- Example 4 converged in 9 iterations
- Example 5 converged in 2 iterations
- Example 6 had not converged after 100 iterations, but if q was reset to the last upper bound, it converged in a further 14 iterations

All convergence was to the correct 4 decimal eigenvalue.

4 Conclusion

This paper presents an always convergent method for finding the spectral radius of a non-negative, irreducible matrix. It is a method closely related to the Power Method and the Inverse Power Method. However, it has advantages over both of these methods, viz certainty of convergence, a reliable estimate of the error at each step and the ability to restart the iterations in the case of very slow convergence. If the matrix A is reducible then converging bounds do not necessarily occur. However, this can be overcome by

adding to A the matrix

$$E = \begin{bmatrix} 0 & \varepsilon & 0 \cdots & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & & \ddots & \varepsilon \\ \varepsilon & 0 & \cdots & 0 \end{bmatrix},$$

for small $\varepsilon > 0$. This ensures that $A + E$ is irreducible, and the method of this paper can be applied. An attendant difficulty is then whether the perturbation in matrix A significantly affects the spectral radius. To determine the impact of this perturbation a result from Stewart [6, p.296] is helpful. This result states that if λ is a simple eigenvalue of A with right eigenvector x and left eigenvector y, with $\|x\|_2 = 1$ and $y^T x = 1$, and A is deflated using an orthogonal matrix R such that

$$R^T A R = \begin{bmatrix} \lambda & h^T \\ 0 & C \end{bmatrix}, \quad \text{then} \quad |\lambda - \lambda'| \leq \varepsilon \|y\|_2 + \frac{\varepsilon^2}{\delta} + \eta\, \mathcal{O}(\varepsilon^2),$$

where λ' is the corresponding eigenvalue of the perturbed matrix $A+E$. Also, $\varepsilon = \|E\|_2$, $\delta = \|(\lambda I - C)^{-1}\|_2^{-1}$ and $\eta = \|h\|_2$. So the numbers $\|y\|_2$, δ and η give a measure of the condition of the simple eigenvalue λ. These numbers were calculated for the dominant eigenvalue of 5000 randomly generated non-negative matrices for each of the orders 5, 10, 20, 50, 100 and 200. In all but a very few exceptional cases, $\|y\|_2$ was close to 1, δ was greater than 0.1 and η less than 1, indicating that typically, the dominant eigenvalue of a non-negative matrix is not greatly affected if the values in E are appropriately small. Computational time is also an important factor in any method and experiments have shown that, for matrices of dimension less than 200, the `eig` routine in MATLAB is faster; but beyond this dimension, the method of this paper appears to be faster. This research is still proceeding and it is intended to compare the method presented here with Krylov subspace methods. Also it is planned to make time comparisons in a compiled language, namely Fortran.

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