# Combination technique coefficients via error splittings

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**Abstract**

We investigate a new way of choosing combination coefficients for the sparse grid combination technique. Previous work considered choosing coefficients such that the interpolation error of sufficiently smooth functions is minimised. We instead obtain an error bound using an error splitting model of approximation error and seek coefficients which minimise this. With minor modification this approach can also yield extrapolations. There are also potential applications to fault tolerance where new coefficients are required when a solution becomes unavailable due to a fault. We test the approach numerically on a scalar advection problem and compare with classical combinations from the literature.

# Contents



# 1 Introduction

Recently there has been much interest in algorithms for high performance computing which are able to recover from and/or adapt to process failures. Harding [3, 4] showed how the combination technique, which approximates sparse grid solutions, is adapted in the event of process failures by exploiting the redundancies given by the many coarse grid approximations. The recovery involves adapting the combination coefficients to avoid coarse grid approximations that were not computed successfully as a result of process failures. Harding [3, 4] focused on updating the coefficients in a way that satisfies an inclusion/exclusion principle over the hierarchical surpluses and minimises the interpolation error for functions in $H^2_{0,\mathrm{mix}}(\Omega) := \{u \in L^2(\Omega) : u|_{\partial\Omega} = 0$ and $D^\alpha u \in L^2(\Omega)$ for all $0 \leqslant \alpha \leqslant 2\}$. This is a somewhat optimistic strategy that assumes that the error is dominated by the representation in sparse grid space rather than from accumulation of truncation errors. In many computations this is not the case and thus in this article we consider a different strategy.

Classical studies of the combination technique typically assume an error model referred to as an error splitting [2]. For many problems, such models are derived via multi-variate Taylor series expansions of the local truncation

error. For example, finite difference and finite volume schemes can typically be analysed in this manner [7, 8, e.g.]. Section 2 introduces error splitting models and discusses how they are typically applied to the combination technique. When accumulation of truncation error is the dominant source of error, and the error splitting model is a good fit, then choosing coefficients based on this model seems a reasonable approach. Section 3 shows how combination coefficients are chosen to minimise a bound on the combination of the error splitting terms. The problem of fitting the model to a given problem is also briefly discussed. Section 4 tests this approach on a simple advection PDE in two dimensions and we compare these results with classical combinations, combinations of multi-variate extrapolations, and combinations based on interpolation estimates.

## 2 The combination technique and error splittings

The combination technique has been studied extensively [2, 5, 1, 8]. Let $\Omega = [0,1]$, for $k \in \mathbb{N}$ we define $h_k := 2^{-k}$ and

$$\Omega_k := \{0, h_k, 2h_k, 3h_k, \dots, (2^k - 1)h_k, 1\}.$$

Let $V_k$ be the space of piecewise linear functions $[0,1] \to \mathbb{R}$ which interpolates between function values given on the points in $\Omega_k$. These function spaces are nested with $V_k \subset V_{k+1}$ for all $k \geqslant 0$. High dimensional approximation spaces are obtained via a tensor product formulation, for $d \geqslant 2$ and $\mathbf{i} = (i_1, \dots, i_d) \in \mathbb{N}^d$ the domain $\Omega^d$ is discretised over the grid points

$$\Omega_{\mathbf{i}} = \Omega_{i_1} \times \cdots \times \Omega_{i_d},$$

leading to piecewise multi-linear function spaces $V_{\mathbf{i}} = V_{i_1} \otimes \cdots \otimes V_{i_d} : \Omega^d \to \mathbb{R}$.

Given a continuous function $u \in V = C(\Omega^d)$ we define $u_{\mathbf{i}} \in V_{\mathbf{i}}$ to be an approximation of $u$. The classical combination technique of level $n$ takes

several such approximations and combines them according to

$$u_n^c := \sum_{k=0}^{d-1} (-1)^k \binom{d-1}{k} \sum_{|\mathbf{i}|=n-k} u_{\mathbf{i}}\,, \tag{1}$$

where $|\mathbf{i}| = i_1 + \cdots + i_d$. This combination approximates level $n$ sparse grid solutions and requires that $u_{\mathbf{i}}$ be computed for $\mathbf{i}$ satisfying $n - d < |\mathbf{i}| \leqslant n$.

The combination technique is adapted to other collections of $\mathbf{i}$, given $I \subset \mathbb{N}^d$ (finite, non-empty) we write

$$u_I = \sum_{\mathbf{i}\in I} c_{\mathbf{i}} u_{\mathbf{i}} \tag{2}$$

for a combination over the multi-indices in I. Of course the challenge here is to determine what the $c_{\mathbf{i}}$ should be to obtain a good approximation to $u$. The consistency property

$$\sum_{\mathbf{i}\in I} c_{\mathbf{i}} = 1\,, \tag{3}$$

must be satisfied to ensure that constant functions, when computed exactly, combine to the exact solution.

When the residual is easily estimated one may find $c_{\mathbf{i}}$ which minimise $\left\|u - \sum_{\mathbf{i}\in I} c_{\mathbf{i}} u_{\mathbf{i}}\right\|$ over an appropriate norm. This is typically referred to as opticom [6]. However, for PDE problems solved via finite difference or finite volume methods the computation of a residual is usually not available. The nature of such methods, whereby stencils are developed and/or analysed via Taylor series expansions, makes error splittings a suitable model for estimating the error. We consider two error splittings from the sparse grid literature [2, 8, 9].

**Definition 1.** *An approximation algorithm* $A_{\mathbf{i}} : V \to V_{\mathbf{i}}$ *for* $\mathbf{i} \in \mathbb{N}^d$ *satisfies an order* $p$ *(point-wise) error splitting if*

$$u - A_{\mathbf{i}}(u) = \sum_{k=1}^{d} \sum_{\substack{\{j_1,\ldots,j_k\}\\ \subset\{1,\ldots,d\}}} \gamma_{j_1,\ldots,j_k}(h_{i_{j_1}}, \ldots, h_{i_{j_k}}) h_{i_{j_1}}^p \cdots h_{i_{j_k}}^p\,, \tag{4}$$

*for all* $x \in \Omega$, $\mathbf{i} \in \mathbb{N}^d$ *with each of the* $\gamma_{j_1,\dots,j_k}(h_{i_{j_1}}, \dots, h_{i_{j_k}})$ *bounded.*

As a point-wise splitting, $u$, $A_{\mathbf{i}}(u)$ and the $\gamma_{j_1,\dots,j_k}$ implicitly depend on $x \in \Omega^d$. Error splittings of this type were used in classical error analysis of the combination technique in two and three dimensions [2] and later extended to all $d \geqslant 2$ dimensions [8]. A similar splitting may also be used to study extrapolation techniques.

**Definition 2.** *An approximation algorithm* $A_{\mathbf{i}} : V \to V_{\mathbf{i}}$ *for* $\mathbf{i} \in \mathbb{N}^d$ *satisfies an order* $p, q$ *(point-wise) error splitting with* $q > p$ *if*

$$u - A_{\mathbf{i}}(u) = \sum_{k=1}^{d} \sum_{\substack{\{j_1,\dots,j_k\} \\ \subset\{1,\dots,d\}}} \Big( \eta_{j_1,\dots,j_k} h_{i_{j_1}}^p \cdots h_{i_{j_k}}^p + \gamma_{j_1,\dots,j_k}(h_{i_{j_1}}, \dots, h_{i_{j_k}}) h_{i_{j_1}}^q \cdots h_{i_{j_k}}^q \Big), \tag{5}$$

*for all* $x \in \Omega$, $\mathbf{i} \in \mathbb{N}^d$ *with each of the* $\eta_{j_1,\dots,j_k}, \gamma_{j_1,\dots,j_k}(h_{i_{j_1}}, \dots, h_{i_{j_k}})$ *bounded.*

In Definition 2 the $\eta$ do not depend on $h_{i_{j_1}}, \dots, h_{i_{j_k}}$. This splitting was used to study the classical combination applied to multi-variate extrapolations as the independence on the spatial discretisation allows the $\eta$ to be eliminated given appropriate combinations [9]. Often $q = p + 2$ is considered where centred spatial discretisations are used. For example, second order centred finite difference stencils eliminate odd order terms of the Taylor series such that $p = 2$ and $q = 4$ is an appropriate error model.

Error splittings can be studied in the combination technique when the consistency property (3) is satisfied as one has

$$u - \sum_{\mathbf{i} \in I} c_{\mathbf{i}} u_{\mathbf{i}} = \sum_{\mathbf{i} \in I} c_{\mathbf{i}} (u - u_{\mathbf{i}}). \tag{6}$$

When the set of indices $I$ is given and the $c_{\mathbf{i}}$ known, an error splitting may be substituted into the right hand side of (6) to study the error. By re-arranging,

eliminating terms where possible and bounding remaining terms, a point-wise error bound is calculated in a manner similar to the classical analysis [2].

# 3 Coefficients via error splittings

Rather than use the error splitting model to estimate the error for given coefficients $\mathbf{i} \in I \subset \mathbb{N}^d$, we instead use the error splitting model to determine the best combination coefficients. In particular we wish to minimise $\left|\sum_{\mathbf{i}\in I} c_{\mathbf{i}}(u - u_{\mathbf{i}})\right|$ subject to the consistency property (3). If $u_{\mathbf{i}} = A_{\mathbf{i}}(u)$ where $A_{\mathbf{i}}$ satisfies the order $p$ error splitting, then

$$\left|\sum_{\mathbf{i}\in I} c_{\mathbf{i}}(u - u_{\mathbf{i}})\right| = \left|\sum_{\mathbf{i}\in I} c_{\mathbf{i}} \sum_{k=1}^{d} \sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}} \gamma_{j_1,\dots,j_k}(h_{i_{j_1}},\dots,h_{i_{j_k}}) h_{i_{j_1}}^p \cdots h_{i_{j_k}}^p\right| \tag{7}$$

$$\leqslant \sum_{k=1}^{d} \sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}} \left|\sum_{\mathbf{i}\in I} c_{\mathbf{i}} \gamma_{j_1,\dots,j_k}(h_{i_{j_1}},\dots,h_{i_{j_k}}) h_{i_{j_1}}^p \cdots h_{i_{j_k}}^p\right|. \tag{8}$$

Minimising (7) directly is difficult because the result is sensitive to the $\gamma_{j_1,\dots,j_k}(h_{i_{j_1}},\dots,h_{i_{j_k}})$ which are typically not known, vary over the domain, and may be difficult to estimate. Instead we look towards minimising something that bounds this term. In a classical combination, many of the $\gamma_{j_1,\dots,j_k}$ terms with $k < d$ cancel, in particular those with $i_{j_1} + \cdots + i_{j_k} \leqslant n - k$. It is therefore reasonable to expect that coefficients which minimise the summation of error splittings would also involve the cancellation of many of these terms, and thus our minimisation problem should facilitate this.

Given $1 \leqslant k \leqslant d$ and $\{j_1,\dots,j_k\} \subset \{1,\dots,d\}$ we define $I_{j_1,\dots,j_k} := \{\boldsymbol{\ell} \in \mathbb{N}^k : \boldsymbol{\ell} = (i_{j_1},\dots,i_{j_k}) \text{ for some } \mathbf{i} \in I\}$. Similarly, given $\boldsymbol{\ell} \in \mathbb{N}^k$ we define $I_{\boldsymbol{\ell}|j_1,\dots,j_k} := \{\mathbf{i} \in I : (i_{j_1},\dots,i_{j_k}) = (\ell_1,\dots,\ell_k)\}$. With this notation we write

the right hand side of (8) as

$$\sum_{k=1}^{d} \sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}} \left| \sum_{\ell\in I_{j_1,\dots,j_k}} \gamma_{j_1,\dots,j_k}(h_{\ell_1},\dots,h_{\ell_k}) h_{\ell_1}^p \cdots h_{\ell_k}^p \sum_{i\in I_{\ell|j_1,\dots,j_k}} c_i \right|$$

$$\leqslant \sum_{k=1}^{d} \sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}} \sum_{\ell\in I_{j_1,\dots,j_k}} |\gamma_{j_1,\dots,j_k}(h_{\ell_1},\dots,h_{\ell_k})| h_{\ell_1}^p \cdots h_{\ell_k}^p \left| \sum_{i\in I_{\ell|j_1,\dots,j_k}} c_i \right| . \quad (9)$$

We now consider the problem of finding $c_i$ which minimise (9). In this form the problem is equivalent to minimising each of the $\left|\sum_{i\in I_{\ell|j_1,\dots,j_k}} c_i\right|$ weighted by the $|\gamma_{j_1,\dots,j_k}(h_{\ell_1},\dots,h_{\ell_k})| h_{\ell_1}^p \cdots h_{\ell_k}^p$ terms. We write this generically as an $L_1$ minimisation problem

$$\text{minimize} \quad \|Wc\|_1 \quad \text{subject to} \quad 1^T c = 1 , \quad (10)$$

where $c$ is a vector of the $\{c_i\}_{i\in I}$ with length $n = |I|$ and $W$ is a $n \times m$ matrix with $m = \sum_{k=1}^{d} \sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}} |I_{j_1,\dots,j_k}|$. We solve this using GLPK[1] via the equivalent linear programming problem

$$\text{minimize} \quad \|d\|_1 \quad \text{subject to} \quad \begin{bmatrix} W & -I \\ -W & -I \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \leqslant \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{with} \quad \|c\|_1 = 1 ,$$

where $I$ is an $m \times m$ identity matrix and $d$ is an $m$-vector.

An interesting problem is the estimation of each $|\gamma_{j_1,\dots,j_k}(h_{\ell_1},\dots,h_{\ell_k})|$ term. This is clearly problem dependant and depends on both $u$ and the algorithm $A_i$ used to approximate $u$. In practice, by comparing several $A_i(u)$ one may obtain rough estimates of such terms. For example, given a two dimensional problem one has

$$\begin{aligned} A_{(i_1,i_2)}(u) - A_{(i_1-1,i_2)}(u) &= [u - A_{(i_1-1,i_2)}(u)] - [u - A_{(i_1,i_2)}(u)] \\ &= \gamma_1(h_{i_1-1}) h_{i_1-1}^p - \gamma_1(h_{i_1}) h_{i_1}^p \\ &\quad + \gamma_{1,2}(h_{i_1-1}, h_{i_2}) h_{i_1-1}^p h_{i_2}^p - \gamma_{1,2}(h_{i_1}, h_{i_2}) h_{i_1}^p h_{i_2}^p . \end{aligned}$$

[1] gnu.org/software/glpk

By increasing $i_2$ such that the $\gamma_{1,2}$ terms are negligible, and assuming that $\gamma_1(h_{i_1}) \approx \gamma_1(h_{i_1-1})$ for sufficiently large $i_1$, one is able to estimate $\gamma_1$. The term $\gamma_2$ is estimated in a similar fashion and $\gamma_{1,2}$ is estimated via

$$A_{(i_1,i_2)}(u) - A_{(i_1-1,i_2)}(u) - A_{(i_1,i_2-1)}(u) + A_{(i_1-1,i_2-1)}(u)\,.$$

This can be extended to higher dimensions. There are times when the elements of $\mathbf{i}$ may not be sufficiently large to accurately estimate the $\gamma_{j_1,\dots,j_k}$ terms, for example in high dimensions where it is too costly to compute $\mathbf{i}$ without some of the components being small. Thus, in this article we also experiment with generically setting each element of the $\gamma_{j_1,\dots,j_k}$ to one.

When $u_i = A_i(u)$ satisfies the order $p, q$ splitting the error bounds over the $\eta$ and $\gamma$ terms are considered separately. In particular,

$$\begin{aligned}\left|\sum_{i\in I} c_i(u-u_i)\right| \leqslant &\left|\sum_{k=1}^{d}\sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}}\sum_{i\in I} c_i\eta_{j_1,\dots,j_k}h_{i_{j_1}}^p\cdots h_{i_{j_k}}^p\right| \\ &+\left|\sum_{k=1}^{d}\sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}}\sum_{i\in I} c_i\gamma_{j_1,\dots,j_k}(h_{i_{j_1}},\dots,h_{i_{j_k}})h_{i_{j_1}}^q\cdots h_{i_{j_k}}^q\right|.\end{aligned}$$

The $\gamma_{j_1,\dots,j_k}(h_{i_{j_1}},\dots,h_{i_{j_k}})$ are bound in the same way as the inequalities (8) and (9) (replacing $p$ with $q$). For the $\eta_{j_1,\dots,j_k}$,

$$\left|\sum_{k=1}^{d}\sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}}\sum_{i\in I} c_i\eta_{j_1,\dots,j_k}h_{i_{j_1}}^p\cdots h_{i_{j_k}}^p\right| \leqslant \sum_{k=1}^{d}\sum_{\substack{\{j_1,\dots,j_k\}\\ \subset\{1,\dots,d\}}}|\eta_{j_1,\dots,j_k}|\left|\sum_{i\in I} c_i h_{i_{j_1}}^p\cdots h_{i_{j_k}}^p\right|.$$

This introduces $2^d-1$ additional terms to the minimisation problem (corresponding to each of the $\eta_{j_1,\dots,j_k}$) which has the same generic formulation as that of (10) but now with $m = 2^d - 1 + \sum_{k=1}^{d}\sum_{\{j_1,\dots,j_k\}\subset\{1,\dots,d\}}|I_{j_1,\dots,j_k}|$. The idea behind including these additional terms is that the coefficients generated by the optimisation are typically such that these terms will sum to zero and therefore the order $p$ terms vanish, resulting in an order $q$ approximation.

# 4 Numerical results

We focus on two dimensional problems, testing the approach on a few different index sets $J \subset \mathbb{N}^2$. Our initial index sets are

$$J_{n,\tau,l} := \{\mathbf{i} \in \mathbb{N}^2 : i_1, i_2 \geqslant \tau \text{ and } n - l < |\mathbf{i}| \leqslant n\},$$

with $n$, $\tau$ and $l$ the level, truncation and layer count parameters, respectively. We require $n \geqslant 2\tau$ for this set to be non-empty. The classical combination coefficients for $\mathbf{i} \in J_{n,\tau,l}$ with $l \geqslant 2$ are

$$c_{\mathbf{i}} = \begin{cases} 1 & \text{if } |\mathbf{i}| = n, \\ -1 & \text{if } |\mathbf{i}| = n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

These combinations are compared with those using $c_{\mathbf{i}}$ derived from the minimisation of (9). We also compute a combination of multi-variate extrapolations for second order schemes [9] which for $\mathbf{i} \in J_{n,\tau,l}$ with $l \geqslant 4$ and $n \geqslant 2(\tau + 2)$ has coefficients

$$c_{\mathbf{i}} = \begin{cases} \frac{16}{9} & \text{if } |\mathbf{i}| = n \text{ and } \mathbf{i} \geqslant (\tau + 1, \tau + 1), \\ \frac{-24}{9} & \text{if } |\mathbf{i}| = n - 1 \text{ and } \mathbf{i} \geqslant (\tau + 1, \tau + 1), \\ \frac{-4}{9} & \text{if } |\mathbf{i}| = n - 1 \text{ and } \mathbf{i} \not\geqslant (\tau + 1, \tau + 1), \\ 1 & \text{if } |\mathbf{i}| = n - 2 \text{ and } \mathbf{i} \geqslant (\tau + 1, \tau + 1), \\ \frac{5}{9} & \text{if } |\mathbf{i}| = n - 2 \text{ and } \mathbf{i} \not\geqslant (\tau + 1, \tau + 1), \\ \frac{-1}{9} & \text{if } |\mathbf{i}| = n - 3, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

By $\mathbf{i} \geqslant (\tau + 1, \tau + 1)$ we mean $i_1 \geqslant \tau + 1$ and $i_2 \geqslant \tau + 1$. Similarly, $\mathbf{i} \not\geqslant (\tau+1, \tau+1)$ means $\mathbf{i} \geqslant (\tau+1, \tau+1)$ and $\mathbf{i} \neq (\tau+1, \tau+1)$. The combination using the coefficients of equation (11) are compared with combinations where the $c_{\mathbf{i}}$ are derived from the minimisation of the order $p, q$ error splitting.

Following this we look at several different randomly chosen subsets $J \subset J_{16,4,9}$ with $\mathbb{E}[|J|] = 0.8|J_{16,4,9}|$ (with $\mathbb{E}[|J|]$ the expectation of the number of elements in $J$) and compare the combinations obtained by minimising the error splitting estimates with those based on minimising interpolation estimates, as shown by Harding et al. [4].

Our tests are performed on a simple advection problem

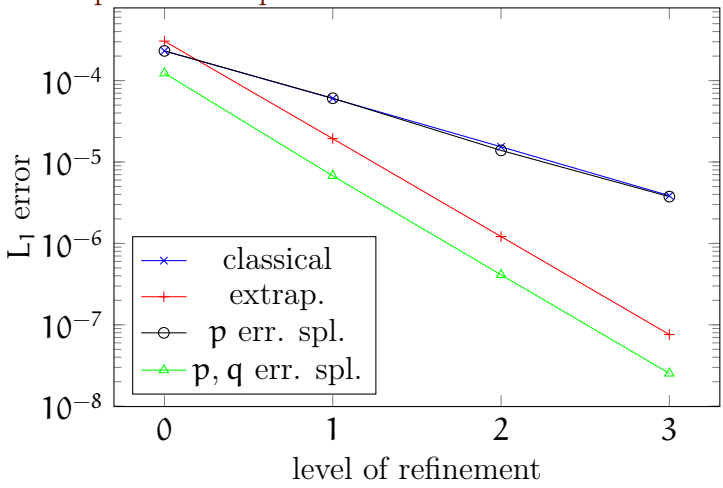
$$\frac{\partial u}{\partial t} + 1^T \cdot \nabla u = 0\,,$$

for $u : [0,1]^2 \to \mathbb{R}$ with initial condition $u_0 = \cos(2\pi x)\sin(2\pi y)$ and periodic boundary conditions. We evolve up to $t = 0.25$ using second order centred finite difference discretisation of spatial derivatives and the classical fourth order Runge–Kutta scheme for integration over time (thus $p = 2$ and $q = 4$ in the error splitting models).

Figure 1 compares the rate of convergence of several methods starting with the index set $J_{12,4,4}$ and then refining each grid uniformly by a factor of two for subsequent computations (corresponding to index sets $J_{14,5,4}$, $J_{16,6,4}$ and $J_{18,7,4}$). We observe that the classical combination and the order $p$ error splitting result have the same order of convergence (two) and have very similar results in general. The combination of extrapolations and the order $p, q$ error splitting both exhibit a higher rate of convergence (four) and the order $p, q$ error splitting result outperforms the combination of extrapolations by a factor of approximately two. A generic weighting of $|\gamma_{j_1,\ldots,j_k}| = |\eta_{j_1,\ldots,j_k}| = 1$ is used in these tests. For different starting sets the results for order $p, q$ error splitting are not always so nice and the rate of convergence may sometimes drop back to roughly second order. It is currently not clear exactly when or why this occurs but we believe it to be related to the instabilities which are typical in extrapolation methods. However, Figure 1 shows there are ranges in which the extrapolation behaves particularly well.

Table 1 compares the error splitting based coefficients with those obtained via interpolation estimates (referred to as gcp), as done by Harding et al. [4]

Figure 1: Starting with $J_{12,4,4}$ we compare the combinations (1), (11) with combinations computed using coefficients derived from the error splitting estimates. The grids of $J_{12,4,4}$ are refined in both spatial dimensions several times and the computations repeated.




on random subsets of grids. We take a random sample $J \subset J_{16,4,9}$ with each multi-index in $J_{16,4,9}$ appearing in $J$ with probability $0.8$. We then compute coefficients using the different approaches and compare the resulting combinations. On the left of the table the first ten samples use the generic unit weighting $|\gamma_{j_1,\dots,j_k}| = 1$ for the error splitting approach (and similarly for $\eta$). We see that the gcp approach outperforms the order $p$ error splitting coefficients ($p$ split). The order $p, q$ error splitting coefficients ($p, q$ split) have higher order convergence and thus this splitting outperforms the order $p$ splitting results but only outperforms the the gcp approach in seven of ten cases. On the right of the table we have an additional ten samples where the $\gamma_{j_1,\dots,j_k}$ are weighted with a rough estimate of $\|\gamma_{j_1,\dots,j_k}\|_\infty$ for the error splitting approaches (and similarly for $\eta$). This rough estimate leads to significant improvements in the '$p$ split' results which typically yields similar combination coefficients to the gcp approach but outperform in some cases.

Table 1: $L_1$ error for combinations obtained via interpolation (gcp) and error splitting ('p split' and 'p, q split') estimates for the combination error over 20 random samples $J \subset J_{16,4,9}$ with each element having an 80% chance of appearing in J. The first ten samples use the generic unit weighting $|\gamma_{j_1,\ldots,j_k}| = 1$ whilst the last ten samples use an estimate of $\|\gamma_{j_1,\ldots,j_k}\|_\infty$.

| | unit weighting | | | $\|\gamma_{j_1,\ldots,j_k}\|_\infty$ estimate | | |
|---|---|---|---|---|---|---|
| sample | gcp | p split | p, q split | gcp | p split | p, q split |
| 1 | 2.6E−6 | 3.7E−6 | 7.9E−7 | 1.5E−6 | 1.5E−6 | 1.3E−6 |
| 2 | 1.3E-6 | 3.1E−5 | 1.5E−6 | 2.6E−6 | 2.6E−6 | 1.3E−6 |
| 3 | 3.3E−6 | 7.9E−6 | 4.7E−6 | 1.2E−6 | 1.2E−6 | 3.7E−7 |
| 4 | 1.2E−6 | 3.7E−6 | 7.2E−7 | 1.2E−6 | 1.2E−6 | 3.7E−7 |
| 5 | 3.3E−6 | 3.6E−6 | 7.9E−7 | 1.5E−6 | 1.5E−6 | 2.3E−7 |
| 6 | 1.9E−6 | 3.0E−5 | 1.6E−6 | 1.5E−6 | 1.3E−6 | 3.7E−7 |
| 7 | 1.3E−6 | 7.9E−6 | 4.1E−7 | 2.6E−6 | 1.2E−6 | 3.7E−7 |
| 8 | 2.6E−6 | 3.6E−6 | 3.7E−6 | 1.2E−6 | 1.2E−6 | 2.8E−5 |
| 9 | 2.6E−6 | 1.3E−5 | 5.4E−7 | 1.8E−6 | 1.6E−6 | 2.4E−7 |
| 10 | 1.6E−6 | 3.1E−5 | 7.9E−7 | 2.8E−6 | 2.8E−6 | 1.7E−6 |
| mean | 2.2E−6 | 1.3E−5 | 1.5E−6 | 1.8E−6 | 1.6E−6 | 3.4E−6 |
| stdev | 0.8E−6 | 1.2E−5 | 1.5E−6 | 0.7E−6 | 0.62E−6 | 8.6E−6 |

The 'p, q split' results also improve and outperform the other approaches with the exception of one outlier. We conclude that the error splitting based coefficients can outperform the gcp approach when an error splitting is applicable and the $\gamma_{j_1,\ldots,j_k}$ are estimated with reasonable accuracy. Without this estimate of $\gamma_{j_1,\ldots,j_k}$ the error splitting results are mixed and appear to be less robust than the those obtained with the gcp approach.

# Conclusions

A new way to compute combination coefficients based on error splitting models was developed and tested on a simple advection problem. Whilst these initial results are promising there are several aspects of this work which could be investigated in more detail. Table 1 demonstrates that the solution is sensitive to the relative size of the constants $\|\gamma_{j_1,\ldots,j_k}\|_\infty$ in the error splitting model. It would be interesting to study how sensitive the solution is to these constants and determine alternative methods of estimating them. The approach was tested on a simple advection problem for which the error splitting model is known to hold [7]. Evaluating the robustness of the approach on more complex problems and higher dimensional problems will be explored in the future. Experience suggests that whilst this approach may work in some cases, results based on interpolation estimates, as calculated by Harding et al. [4], are more robust.

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