Estimating the error of a H^1-mixed finite element solution for the Burgers equation

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Abstract

We compute error estimations for a H^1-mixed finite element method for Burgers equation. By using a H^1-mixed finite element method, the problem is reformulated as a system of first order partial differential equations, which allows an approximation of the unknown function and its derivative. *Local* parabolic and elliptic methods approximate the true errors from the computed solutions; the so-called a posteriori error estimates. Numerical experiments show that the error estimations converge to the true errors.

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1 Introduction

We consider the Burgers equation

$$\partial_t u(x,t) - \nu \partial_{xx} u(x,t) + u(x,t)\partial_x u(x,t) = 0\,, \quad x \in \Omega\,, \quad t \in (0,T]\,, \tag{1}$$

with boundary and initial conditions

$$u(0,t) = u(1,t) = 0\,, \quad t \in [0,T]\,, \tag{2}$$
$$u(x,0) = u_0(x)\,, \quad x \in \Omega\,, \tag{3}$$

where $\partial_t := \partial/\partial t$, $\partial_x := \partial/\partial x$, $\partial_{xx} := \partial^2/\partial x^2$, T and ν (viscosity coefficient) are positive constants and $\Omega := (0,1)$ [2].

The aim is to design methods to compute a posteriori error estimations when the solution of (1)–(3) is approximated by a H^1-mixed finite element method (H^1-MFEM).

Using the H^1-MFEM, the problem is reformulated as a system of first order partial differential equations, which allows an approximation for u and its derivative $\partial_x u$. The H^1-MFEM considered in this article is based on an approach suggested by Pani for nonlinear parabolic equations [3]. Pany et al. [4] adapted the method to Burgers equation. Section 2 gives details of this H^1-MFEM.

The method considered in this article is closely related to least squares mixed finite element methods in that the second order partial differential equation is reformulated as a system of first order partial differential equations with a new unknown defined as the flux [7, 8, 9, 10, and references therein].

A posteriori error estimates are a fundamental component in the design of efficient adaptive algorithms for solving partial differential equations. In this study we consider an implicit type of a posteriori error estimation which is based on the procedure developed by Adjerid et al. [1] for one dimensional parabolic systems. This a posteriori error estimation with finite element methods of lines was studied for one dimensional nonlinear parabolic system and the Sobolev equation [5, 6]. For the approximation of the solution we use a mixed formulation of finite element methods of lines with an a posteriori error estimation computed using the procedure developed by Adjerid et al. [1]. To the best of our knowledge, this is the first time this a posteriori error estimation method is considered for the Burgers equation, where the approximate solution is computed using H^1-MFEM.

2 The H^1-mixed finite element method

Throughout this article, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_{H^0(\Omega)}$ denote the inner product and norm in $H^0(\Omega) = L^2(\Omega)$, respectively. As usual, the Sobolev space $H^1(\Omega)$ consists of functions u for which

$$\|u\|_{H^1(\Omega)} = \sqrt{\|u\|^2_{H^0(\Omega)} + \|\partial_x u\|^2_{H^0(\Omega)}}$$

exists. The space $H^1_0(\Omega)$ contains all functions in $H^1(\Omega)$ with zero trace at the endpoints of the domain Ω, namely at $x = 0, 1$. For any $p \in [0, \infty]$ and any normed vector space X, $L^p(X)$ is the space $L^p(0,T;X)$ of all functions defined in $[0,T]$ with values in X. The norm in this space $\|\cdot\|_{L^p(X)}$ is defined as usual. We write $L^p(L^\infty)$ and $L^p(H^1)$ instead of $L^p(L^\infty(\Omega))$ and $L^p(H^1(\Omega))$, respectively.

By H^1-MFEM, (1) is reduced to a system of first order equations using a new variable defined as $v = u_x$. As a consequence, (1) is reformulated as

$$\partial_x u = v\,, \tag{4}$$

$$\partial_t u - \nu\partial_x v + uv = 0\,. \tag{5}$$

We multiply (4) by $\partial_x\chi$ and (5) by $-\partial_x w$, where $\chi \in H^1_0(\Omega)$ and $w \in H^1(\Omega)$ are arbitrary test functions. Then, using integration by parts and applying the boundary conditions (2), we obtain a weak formulation of (1)–(3): given $u_0 \in H^1_0(\Omega)$, find $(u,v) : [0,T] \to H^1_0(\Omega) \times H^1(\Omega)$ satisfying, for $t > 0$,

$$\langle \partial_x u(t), \partial_x\chi\rangle = \langle v(t), \partial_x\chi\rangle \quad \text{for all } \chi \in H^1_0(\Omega)\,, \tag{6}$$

$$\langle \partial_t v(t), w\rangle + \nu\,\langle \partial_x v(t), \partial_x w\rangle = \langle u(t)v(t), \partial_x w\rangle \quad \text{for all } w \in H^1(\Omega)\,, \tag{7}$$

and, for $t = 0$,

$$\langle v(0), w\rangle = \langle \partial_x u_0, w\rangle \quad \text{for all } w \in H^1(\Omega)\,. \tag{8}$$

Remark 1. Existence and uniqueness of the solution of (6)–(8) can be shown using the method of compactness [12, 11]. We will present this result in a future paper.

Remark 2. If $u \in W^1_\infty(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$ and (u,v) satisfies (6)–(7) then (u,v) satisfies (4)–(5). Indeed, by using integration by parts we deduce from (6) that $\partial_x(v - \partial_x u) = 0$ which implies

$$v(x,t) = \partial_x u(x,t) + g(t)\,, \tag{9}$$

for some function g depending on t. Integrating (9) over Ω, noting (8), we infer $g(0) = 0$. Also, it follows from (9) and (7) (with $w = 1$) that

$$\int_\Omega [\partial_{tx} u + g'(t)]\,dx = 0\,, \tag{10}$$

implying $g'(t) = 0$. Hence $g \equiv 0$, that is (u, v) satisfies (4). This immediately gives (5).

Solutions to (6)–(8) are approximated using a high order finite element method defined as follows. We first partition the interval Ω into $0 = x_1 < x_2 < \cdots < x_{N+1} = 1$, and define $h_l := x_{l+1} - x_l$ for $l = 1, \ldots, N$ and $h := \max_l h_l$. The hat function on (x_{l-1}, x_{l+1}) for $l = 2, \ldots, N$ is defined as

$$\phi_{l,1}(x) = \begin{cases} (x - x_{l-1})/h_{l-1}\,, & x \in [x_{l-1}, x_l)\,, \\ (x_{l+1} - x)/h_l\,, & x \in [x_l, x_{l+1})\,, \\ 0, & \text{otherwise.} \end{cases}$$

At the endpoints of Ω (namely, at $x = 0, 1$) we define

$$\phi_{1,1}(x) = \begin{cases} (x_2 - x)/h_1\,, & x \in [x_1, x_2)\,, \\ 0, & \text{otherwise,} \end{cases}$$

$$\phi_{N+1,1}(x) = \begin{cases} (x - x_N)/h_N\,, & x \in [x_N, x_{N+1})\,, \\ 0, & \text{otherwise.} \end{cases}$$

The space of piecewise linear functions on Ω and its subspace consisting of functions vanishing at the endpoints of Ω are, respectively,

$$\mathcal{S}_h := \operatorname{span}\{\phi_{1,1}, \phi_{2,1} \ldots, \phi_{N+1,1}\} \quad \text{and} \quad \mathring{\mathcal{S}}_h := \operatorname{span}\{\phi_{2,1}, \ldots, \phi_{N,1}\}.$$

The spaces of bubble functions in Ω are defined by $\mathcal{S}_h^k := \operatorname{span}\{\phi_{1,k}, \ldots, \phi_{N,k}\}$, where $\phi_{l,k}$ is an antiderivative of the Legendre polynomial P_{k-1} of degree $k-1$ scaled to the subinterval $[x_l, x_{l+1}]$. More precisely, for $l = 1, \ldots, N$ and $k = 2, 3, \ldots$, we define

$$\phi_{l,k}(x) = \begin{cases} \left[\sqrt{2(2k-1)}/h_l\right] \int_{x_l}^{x} P_{k-1}(y)\,dy\,, & x \in [x_l, x_{l+1})\,, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

For $p, q \in \mathbb{N}$ and $p, q \geqslant 2$, the finite dimensional subspaces of $H^1(\Omega)$ and $H^1_0(\Omega)$ are, respectively,

$$\mathcal{V}_h^q := \mathcal{S}_h \cup \sum_{k=2}^{q} \mathcal{S}_h^k \quad \text{and} \quad \mathring{\mathcal{V}}_h^p := \mathring{\mathcal{S}}_h \cup \sum_{k=2}^{p} \mathcal{S}_h^k .$$

A semidiscrete approximation to (6)–(8) is to find $(U, V) : [0, T] \to \mathring{\mathcal{V}}_h^p \times \mathcal{V}_h^q$ such that for $t \in (0, T]$:

$$\langle \partial_x U(t), \partial_x \chi_h \rangle = \langle V(t), \partial_x \chi_h \rangle , \quad \text{for all } \chi_h \in \mathring{\mathcal{V}}_h^p , \tag{12}$$

$$\langle \partial_t V(t), w_h \rangle + \nu \langle \partial_x V(t), \partial_x w_h \rangle = \langle UV(t), \partial_x w_h \rangle , \quad \text{for all } w_h \in \mathcal{V}_h^q , \tag{13}$$

and

$$\langle V(0), w_h \rangle = \langle \partial_x u_0, w_h \rangle \quad \text{for all } w_h \in \mathcal{V}_h^q . \tag{14}$$

Let the errors in the approximation of (6)–(8) by (12)–(14) be $e(x, t) := u(x, t) - U(x, t)$ and $f(x, t) := v(x, t) - V(x, t)$. This leads to Proposition 3, the proof of which will be presented in a future paper.

Proposition 3. *Assume that* $u, \partial_t u \in L^\infty(H^1_0(\Omega) \cap H^{p+1}(\Omega))$ *and* $v, \partial_t v \in L^\infty(H^{q+1}(\Omega))$. *Assume further that* $U \in L^\infty(\mathring{\mathcal{V}}_h^p)$ *and* $V \in L^\infty(\mathcal{V}_h^q)$. *Then, there exist positive constant* $C > 0$ *independent of* h *such that*

$$\|e(t)\|_j \leqslant Ch^{\min(p+1-j, q+1)} \left[\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^2(H^{q+1})} \right] ,$$

$$\|f(t)\|_j \leqslant Ch^{\min(p+1, q+1-j)} \left[\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^2(H^{q+1})} \right] .$$

Now we show the computation of (U, V). With $\phi_{l,k}$ defined by (11), the solutions to (12)–(14) are

$$U(x, t) = \sum_{l=2}^{N} U_{l,1}(t) \phi_{l,1}(x) + \sum_{l=1}^{N} \sum_{k=2}^{p} U_{l,k}(t) \phi_{l,k}(x) , \tag{15}$$

$$V(x, t) = \sum_{l=1}^{N+1} V_{l,1}(t) \phi_{l,1}(x) + \sum_{l=1}^{N} \sum_{k=2}^{q} V_{l,k}(t) \phi_{l,k}(x) . \tag{16}$$

Let

$$\alpha_{k,k'}^{l,l'} = \langle \phi_{l,k}, \phi_{l',k'} \rangle\,, \quad \bar{\alpha}_{k,k'}^{l,l'} = \langle \partial_x \phi_{l,k}, \partial_x \phi_{l',k'} \rangle\,, \quad \beta_{k,k'}^{l,l'} = \langle \phi_{l,k}, \partial_x \phi_{l',k'} \rangle\,. \tag{17}$$

For each $l = 1, \ldots, N$ and $r, r' = 2, 3, \ldots$ we define a 2×2 matrix $M_{1,1}^l$, a $2 \times (r-1)$ matrix $M_{1,r}^l$, and an $(r-1) \times (r'-1)$ matrix $M_{r,r'}^l$ with entries

$$\begin{aligned} [M_{1,1}^l]_{ij} &= \alpha_{1,1}^{l+j-1,l+i-1}, && i,j = 1,2\,, \\ [M_{1,r}^l]_{ij} &= \alpha_{j,1}^{l,l+i-1}, && i = 1,2\,, j = 2, \ldots, r\,, \\ [M_{r,r'}^l]_{ij} &= \alpha_{j,i}^{l,l}\,, && i = 2, \ldots, r\,, j = 2, \ldots, r'\,. \end{aligned}$$

Similarly, we define matrices $S_{1,1}^l$, $S_{1,r}^l$, $S_{r,r'}^l$ with $\bar{\alpha}_{r,r}^{l,l'}$, and $B_{1,1}^l$, $B_{1,r}^l$, $B_{r,r'}^l$ with $\beta_{r,r}^{l,l'}$. We then define

$$M_r^l = \begin{bmatrix} M_{1,1}^l & M_{1,r}^l \\ (M_{1,r}^l)^T & M_{rr}^l \end{bmatrix}, \quad S_r^l = \begin{bmatrix} S_{1,1}^l & S_{1,r}^l \\ (S_{1,r}^l)^T & S_{rr}^l \end{bmatrix},$$
$$B_{r,r'}^l = \begin{bmatrix} B_{1,1}^l & B_{1,r'}^l \\ (B_{1,r}^l)^T & B_{r,r'}^l \end{bmatrix}.$$

The matrices M_r^l and S_r^l have size $(r+1) \times (r+1)$, whereas the matrix $B_{r,r'}^l$ has size $(r+1) \times (r'+1)$. The global matrices M_r, S_r and $B_{r,r'}$ have elements M_r^l, S_r^l and $B_{r,r'}^l$, respectively. The sizes of M_r and S_r are $(Nr+1) \times (Nr+1)$ and the size of $B_{r,r'}$ is $(Nr+1) \times (Nr'+1)$.

For each $l = 1, \ldots, N$ we also define vectors

$$\mathbf{U}^l = [U_{l,1}, U_{l+1,1}, U_{l,2} \ldots, U_{l,p}]^T \quad \text{and} \quad \mathbf{V}^l = [V_{l,1}, V_{l+1,1}, V_{l,2} \ldots, V_{l,q}]^T\,,$$

where the elements are defined in (15)–(16) and $U_{1,1}$ and $U_{N+1,1}$ are zero. The vectors $\mathbf{U}$ and $\mathbf{V}$ are of size $(Np+1) \times 1$ and $(Nq+1) \times 1$, respectively, and are assembled from the vectors $\mathbf{U}^l$ and $\mathbf{V}^l$.

With the matrices defined above, the matrix representation of (12)–(13) is

$$S_p\mathbf{U}(t) = B_{p,q}\mathbf{V}(t)\,, \tag{18}$$

$$M_q\partial_t\mathbf{V}(t) + \nu S_q\mathbf{V}(t) = \mathbf{G}[\mathbf{U}(t), \mathbf{V}(t)]\,. \tag{19}$$

Here,

$$\mathbf{G}(\mathbf{U}, \mathbf{V}) = [\mathbf{G}^{(0)}, \mathbf{G}^{(1)}, \ldots, \mathbf{G}^{(N)}]^T$$

is an $(Nq+1)\times 1$ vector with

$$\mathbf{G}^{(0)} = [\langle \mathbf{U}\mathbf{V}, \phi_{1,1}\rangle\,, \langle \mathbf{U}\mathbf{V}, \phi_{2,1}\rangle \ldots, \langle \mathbf{U}\mathbf{V}, \phi_{N+1,1}\rangle]^T$$

and

$$\mathbf{G}^{(l)} = [\langle \mathbf{U}\mathbf{V}, \phi_{l,2}\rangle\,, \langle \mathbf{U}\mathbf{V}, \phi_{l,3}\rangle \ldots, \langle \mathbf{U}\mathbf{V}, \phi_{l,q}\rangle]^T$$

for $l = 1, \ldots, N$. We use the Matlab ODE solver to solve (18)–(19). The right hand side of (19) is computed by first solving (18) for a given $\mathbf{V}(t)$.

3 A posteriori error estimates and implementation issues

In this section we design methods to compute the error estimates. We infer that e and f satisfy

$$\langle \partial_x e, \partial_x \chi_h\rangle = \langle f, \partial_x\chi_h\rangle \quad \text{for all } \chi_h \in \mathring{\mathcal{V}}_h^p\,, \tag{20}$$

$$\begin{aligned} &\langle \partial_t f, w_h\rangle + \nu\,\langle \partial_x f, \partial_x w_h\rangle - \langle ef, \partial_x w_h\rangle - \langle Uf, \partial_x w_h\rangle - \langle eV, \partial_x w_h\rangle \\ &= -\nu\,\langle \partial_x V, \partial_x w_h\rangle + \langle UV, \partial_x w_h\rangle - \langle \partial_t V, w_h\rangle \quad \text{for all } w_h \in \mathcal{V}_h^q\,. \end{aligned} \tag{21}$$

At $t = 0$, from (8) and (14),

$$\langle f, w_h\rangle = 0 \quad \text{for all } w_h \in \mathcal{V}_h^q\,.$$

Due to (13), the right hand side of (21) vanishes. However, for the purpose of developing a posteriori error estimates, we keep these terms in the equation. We approximate the exact errors e and f, respectively, by

$$E(x,t) = \sum_{l=1}^{N} E_l(t)\phi_{l,p+1}(x) \in \mathcal{S}_h^{p+1},$$
$$F(x,t) = \sum_{l=1}^{N} F_l(t)\phi_{l,q+1}(x) \in \mathcal{S}_h^{q+1}.$$

which are computed locally on each element (x_l, x_{l+1}), for $l = 1, \dots, N$, from the approximate solutions (U, V).

An accurate error estimation is one that satisfies

$$\lim_{h\to 0} \Theta(t) = 1\,, \quad t \in [0,T]\,, \tag{22}$$

where

$$\Theta(t) := \frac{\hat{E}(t)}{\hat{e}(t)}\,,$$

with

$$\hat{e}(t) := \|e(t)\|_{H^1(\Omega)} + \|f(t)\|_{H^1(\Omega)}\,, \quad \hat{E}(t) := \|E(t)\|_{H^1(\Omega)} + \|F(t)\|_{H^1(\Omega)}\,.$$

We propose four different methods to compute E_l and F_l, $l = 1, \dots, N$. The first equation to be solved for each method is:

1. *Nonlinear parabolic error estimate:* (cf. (21))

$$\begin{aligned}&\langle \partial_t F_l, \phi_{l,q+1}\rangle_l + \nu\,\langle \partial_x F_l, \partial_x \phi_{l,q+1}\rangle_l - \langle E_l F_l, \partial_x \phi_{l,q+1}\rangle_l - \langle U F_l, \partial_x \phi_{l,q+1}\rangle_l \\ &- \langle V E_l, \partial_x \phi_{l,q+1}\rangle_l \\ &= -\nu\,\langle \partial_x V, \partial_x \phi_{l,q+1}\rangle_l + \langle U V, \partial_x \phi_{l,q+1}\rangle_l - \langle \partial_t V, \phi_{l,q+1}\rangle_l\,.\end{aligned}$$

2. *Nonlinear elliptic error estimate:* We neglect the time rate of change in Method 1 so that

$$\begin{aligned}&\nu\left\langle\partial_x F_l,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle E_l F_l,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle U F_l,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle V E_l,\partial_x\phi_{l,q+1}\right\rangle_l\\&=-\nu\left\langle\partial_x V,\partial_x\phi_{l,q+1}\right\rangle_l+\left\langle U V,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle\partial_t V,\phi_{l,q+1}\right\rangle_l .\end{aligned}$$

3. *Linear parabolic error estimate:* An additional reduction in the computation cost is obtained by neglecting the nonlinear term $\left\langle E_l F_l,\partial_x\phi_{l,q+1}\right\rangle_l$ in Method 1:

$$\begin{aligned}&\left\langle\partial_t F_l,\phi_{l,q+1}\right\rangle_l+\nu\left\langle\partial_x F_l,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle U F_l,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle V E_l,\partial_x\phi_{l,q+1}\right\rangle_l\\&=-\nu\left\langle\partial_x V,\partial_x\phi_{l,q+1}\right\rangle_l+\left\langle U V,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle\partial_t V,\phi_{l,q+1}\right\rangle_l .\end{aligned}\tag{23}$$

4. *Linear elliptic error estimate:* We neglect the nonlinear term in Method 2 so that

$$\begin{aligned}&\nu\left\langle\partial_x F_l,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle U F_l,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle V E_l,\partial_x\phi_{l,q+1}\right\rangle_l\\&=-\nu\left\langle\partial_x V,\partial_x\phi_{l,q+1}\right\rangle_l+\left\langle U V,\partial_x\phi_{l,q+1}\right\rangle_l-\left\langle\partial_t V,\phi_{l,q+1}\right\rangle_l .\end{aligned}$$

Each equation in Methods 1–4 is coupled with (cf. (20))

$$\left\langle\partial_x E_l,\partial_x\phi_{l,p+1}\right\rangle_l=\left\langle F_l,\partial_x\phi_{l,p+1}\right\rangle_l ,\tag{24}$$

and an initial condition $\left\langle F_l,\phi_{l,q+1}\right\rangle=\left\langle\partial_x u_0,\phi_{l,q+1}\right\rangle-\left\langle V,\phi_{l,q+1}\right\rangle$.

We finish this section with a discussion on implementation issues for the linear parabolic case. From (17), we have

$$\left\langle\partial_x V,\partial_x\phi_{l,q+1}\right\rangle_l=V_{l+1,1}\bar\alpha^{l+1,l}_{1,q+1}+\sum_{k'=1}^{q}V_{l,k'}\bar\alpha^{l,l}_{k',q+1}:=T_1 ,$$

and

$$\left\langle\partial_t V,\phi_{l,q+1}\right\rangle_l=\partial_t V_{l+1,1}\alpha^{l+1,l}_{1,q+1}+\sum_{k'=1}^{q}\partial_t V_{l,k'}\alpha^{l,l}_{k',q+1}:=T_2 .$$

Also

$$\alpha^{l,l}_{p+1,p+1} = h_l/((2p+3)(2p-1))\,, \quad \bar{\alpha}^{l,l}_{p+1,p+1} = 2/h_l\,,$$

and

$$\beta^{l,l}_{p+1,q+1} = \begin{cases} 1/\sqrt{(2q+3)(2q+1)}\,, & p = q+1\,,\\ -1/\sqrt{(2q+1)(2q-1)}\,, & p = q-1\,,\\ 0, & \text{otherwise.}\end{cases}$$

By defining

$$\bar{\beta}^{l}_{k,k',q} = \left\langle \phi_{l,\bar{k}}\phi_{l,k'}, \partial_x \phi_{l,q+1}\right\rangle, \quad \tilde{\beta}^{l}_{k,k',q} = \left\langle \phi_{l+1,\bar{k}}\phi_{l,k'}, \partial_x \phi_{l,q+1}\right\rangle$$

and

$$\hat{\beta}^{l}_{k,k',q} = \left\langle \phi_{l+1,\bar{k}}\phi_{l+1,k'}, \partial_x \phi_{l,q+1}\right\rangle,$$

we have

$$\langle UF_l, \partial_x \phi_{l,q+1}\rangle_l = F_l \left[U_{l+1,1}\tilde{\beta}^{l}_{1,q+1,q} + \sum_{k=1}^{p} U_{l,k}\bar{\beta}^{l}_{k,q+1,q}\right] := T_3 F_l\,,$$

$$\langle VE_l, \partial_x \phi_{l,q+1}\rangle_l = E_l \left[V_{l+1,1}\tilde{\beta}^{l}_{1,p+1,q} + \sum_{k'=1}^{q} V_{l,k'}\bar{\beta}^{l}_{k',p+1,q}\right] := T_4 E_l\,,$$

and

$$\begin{aligned}\langle UV, \partial_x \phi_{l,q+1}\rangle_l = U_{l+1,1}&\left[V_{l+1,1}\hat{\beta}^{l}_{1,1,q} + \sum_{k'=1}^{q} V_{l,k'}\bar{\beta}^{l}_{1,k',q+1}\right]\\ &+ \sum_{k=1}^{p} U_{l,k}\left[V_{l+1,1}\hat{\beta}^{l}_{k,1,q} + \sum_{k'=1}^{q} V_{l,k'}\bar{\beta}^{l}_{k,k',q}\right] := T_5\,.\end{aligned}$$

The values of $\bar{\beta}^{l}_{k,k',q}$, $\tilde{\beta}^{l}_{k,k',q}$ and $\hat{\beta}^{l}_{k,k',q}$ can be computed using Maple.

By using the above definitions of $T_1, \ldots, T_5$, (23) is rewritten as

$$\frac{h_l}{(2q+3)(2q-1)}\partial_t F_l(t) + \left(\frac{2\nu}{h_l} - T_3\right) F_l(t) - T_4 E_l(t) = -\nu T_1 + T_5 - T_2\,.$$

Moreover, (24) is rewritten as

$$2E_l(t) = h_l \beta^{l,l}_{p+1,q+1} F_l(t)\,.$$

Then, by using the backward Euler formulation, we compute $F_l(t_j)$ recursively using

$$\left(d + \frac{2\nu}{h_l} - T_3 - T_4 \beta^{l,l}_{p+1,q+1} \frac{h_l}{2}\right) F_l(t_j) = -\nu T_1 + T_5 - T_2 + dF_l(t_{j-1})\,, \quad (25)$$

where $d = h_l/(2q+3)(2q-1)(t_j - t_{j-1})$ and $t_j = j\Delta t$ for $j = 1, 2, 3, \ldots$. The time step Δt is chosen to be not less than h.

4 Numerical results

In this section, we present the numerical results obtained when solving (1)–(3) whose exact solutions are

$$u(x,t) = \frac{2\nu\pi a \sin(\pi x)}{2 + a\cos(\pi x)}\,, \quad \nu(x,t) = \frac{2\nu\pi^2 a \cos(\pi x)}{2 + a\cos(\pi x)} + \frac{2\nu[\pi a \sin(\pi x)]^2}{[2 + a\cos(\pi x)]^2}\,,$$

where $a = \exp(-\pi^2 \nu t)$. The initial value is

$$u_0(x) = 2\nu\pi \sin(\pi x)/(2 + \cos(\pi x))\,.$$

In the following, we choose $\nu = 0.05$ and $p = q + 1$. The numerical results are also satisfactory for a larger ν, such as $\nu = 0.5$. We present the numerical results for $\nu = 0.05$ only.

In the numerical experiment, we computed the approximate solution (U, V) by solving (18)–(19). We then computed the errors e and f to check on the convergence rate given by Proposition 3. Finally, we computed the error estimations E and F by using the linear parabolic and linear elliptic a posteriori error estimate methods 3 and 4 introduced in Section 3.

Table 1: The orders of convergence for (u, v) at $t = 0.8$.

p	q	N	$\|e_h(t)\|_{H^1(\Omega)}$	κ_u	$\|f_h(t)\|_{H^1(\Omega)}$	κ_v
2	1	20	1.1338E-3		6.6472E-2	
		40	2.8487E-4	1.993	3.3245E-2	0.999
		80	7.1305E-5	1.998	1.6624E-2	1.000
		160	1.7831E-5	1.999	8.3120E-3	1.000
3	2	20	2.2153E-5		2.8620E-3	
		40	2.7675E-6	3.000	7.1684E-4	1.997
		80	3.4587E-7	3.000	1.7929E-4	1.999
		100	1.7708E-7	3.000	1.1476E-4	1.999

Table 2: The effectivity indexes Θ at $t = 0.8$.

				Method 3		Method 4	
p	q	h	$\hat{e}(t)$	$\hat{E}(t)$	$\Theta(t)$	$\hat{E}(t)$	$\Theta(t)$
2	1	1/20	6.7606E-2	6.6984E-2	0.991	6.6543E-2	0.984
		1/40	3.3530E-2	3.3364E-2	0.995	3.3308E-2	0.993
		1/80	1.6695E-2	1.6652E-2	0.997	1.6645E-2	0.997
		1/160	8.3298E-3	8.3188E-3	0.999	8.3180E-3	0.999
3	2	1/20	2.8841E-3	2.8736E-3	0.996	2.8671E-3	0.994
		1/40	7.1961E-4	7.1833E-4	0.998	7.1792E-4	0.998
		1/80	1.7964E-4	1.7948E-4	0.999	1.7946E-4	0.999
		1/100	1.1493E-4	1.1485E-4	0.999	1.1484E-4	0.999

Table 1 presents the exact errors $\|e(t)\|_{H^1(\Omega)}$ and $\|f(t)\|_{H^1(\Omega)}$ for $t = 0.8$. As predicted by Proposition 3, the convergence rate is $\|e(t)\|_{H^1(\Omega)} = O(h^p)$ and $\|f(t)\|_{H^1(\Omega)} = O(h^{p-1})$. Table 2 presents the computed a posteriori error estimate $\hat{E}$ and the effectivity index $\Theta(t)$, at $t = 0.8$. For Method 3, when solving (25) we chose $\Delta t = 0.4$. The results show that our a posteriori error estimations are efficient.

5 Conclusion

We designed algorithms to estimate the true error when a model problem is solved using H^1-MFEM. Our numerical experiments support our theoretical claims in Proposition 3 and (22). We emphasise that the computation of the error estimations (E_l, F_l) for $l = 1, \dots, N$ can be carried out in parallel on each element (x_l, x_{l+1}). A theoretical study to show $\lim_{h\to 0} \Theta(t) = 1$ is the subject of a future paper.

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