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On the noise-resolution duality, Heisenberg uncertainty and Shannon's information

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Abstract

Several variations of the Heisenberg uncertainty inequality are derived on the basis of 'noise-resolution duality' recently proposed by us. The same approach leads to a related inequality that provides an upper limit for the information capacity of imaging systems in terms of the number of imaging quanta (particles) used in the experiment. These results are useful in the context of biomedical imaging constrained by the radiation dose delivered to the sample, or in imaging (e.g., astronomical) problems under low light conditions.

Contents

1 Introduction

Among the most important characteristics of many imaging, scattering and measuring experimental setups (systems) are the spatial resolution and the signal-to-noise ratio (SNR) [4, 2]. For mainly historical reasons (abundance of photons in typical visible light imaging applications), the two properties are usually considered separately, even though any applied physicist or optical engineer would be aware of an intrinsic link between them. These two characteristics, and the interplay between them, attained additional relevance in recent years in the context of biomedical imaging, where the samples are often sensitive to the radiation dose [15], in certain astronomical methods where the detectable photon flux can be extremely low [10], as well as in some other problems, including those related to the foundations of quantum physics [1]. In x-ray medical imaging, in particular, it is critically important to minimize the radiation dose delivered to the patient, while still being able to obtain 2D or 3D images with sufficient spatial resolution and large enough SNR to detect the features of interest, such as small tumours [13, 14]. In this context, an imaging system (e.g., a CT scanner) must maximize the amount of relevant information extractable from the collected images, while keeping sufficiently low the number of x-ray photons impinging on the patient. This article addresses some mathematical properties of generic imaging systems that are likely to be important in the context of designing the next generation of medical imaging instruments, and may also have relevance to some fundamental aspects of quantum physics and information theory.

Gureyev et al. [13, 14] recently introduced a dimensionless 'intrinsic quality' characteristic Q_S which incorporates both the noise propagation and the spatial resolution properties of a linear shift-invariant (LSI) imaging system:

$$Q_S = \frac{\mathrm{SNR}}{\mathrm{SNR}_{\mathrm{in}}} \left(\frac{\Delta x_{\mathrm{in}}}{\Delta x} \right)^{n/2} = \frac{\mathrm{SNR}}{F_{\mathrm{in}}^{1/2} (\Delta x)^{n/2}} , \tag{1}$$

where n is the dimensionality of the input data ($n = 2$ corresponds to conventional planar images), Δx_{in} and Δx are the input and output spatial resolution of the imaging system, $\mathrm{SNR}_{\mathrm{in}} = S_{\mathrm{in}}/\sigma_{\mathrm{in}}$ and $\mathrm{SNR} = S/\sigma$ are the input and output signal-to-noise ratios, respectively. Gureyev et al. [13, 14] assumed that the incident fluence corresponds to a spatially stationary and uncorrelated random process with Poisson statistics, hence $\mathrm{SNR}_{\mathrm{in}}^2 = F_{\mathrm{in}}(\Delta x_{\mathrm{in}})^n$ is the average number of particles incident on the input spatial resolution unit, where F_{in} is incident particle/quanta fluence (the number of incident particles per n-dimensional volume). Because Q_S is normalised with respect to the incident fluence, it may be viewed as 'imaging quality per single incident particle'. In practice, if the incident fluence rate or the exposure time is increased, then the quality of the resultant image is expected to increase too (normally in proportion to $F_{\mathrm{in}}^{1/2}$) [13, 14]. However, in applications where the imaging quanta are at premium (e.g., in low-light imaging) or where the irradiation dose delivered to the sample is critical (as in x-ray or electron imaging of biological samples), Q_S represents a key performance indicator of the imaging system.

Gureyev et al. [13, 7, 17] showed that when the total number of imaging quanta is fixed, a duality exists between the signal-to-noise and the spatial resolution of the imaging system and, as a result, the intrinsic quality Q_S has an absolute upper limit (maximum):

$$Q_S^2 \leqslant 1/C_n\,, \tag{2}$$

where $C_n = 2^n\Gamma(n/2)n(n+2)/(n+4)^{n/2+1}$ is the Epanechnikov constant [9, 13, 7]. More precisely, de Hoog, Schmalz and Gureyev [7] showed that inequality (2) holds and is exact for LSI systems with point spread functions (PSFs) $T(\mathbf{x})$ with finite mathematical expectation, variance and energy. The maximum is achieved on Epanechnikov PSF $T_E(\mathbf{x}) = (1-|\mathbf{x}|^2)_+$, where the subscript '+' denotes that $T_E(\mathbf{x}) = 0$ at points where the expression in brackets is negative [7]. Section 2 gives further details about this result.

Although the definition of the intrinsic quality was originally introduced for LSI systems [13], it can be extended to some non-linear systems as shown, for example, by Nesterets and Gureyev [17] for the famous Young double-slit diffraction experiment. In that context $F_{in} = N_q/A^n$, where N_q is the total number of incident quanta and A^n is the 'area' of the entrance aperture of the imaging system, so, from equation (1),

$$Q_S^2 = \frac{\mathrm{SNR}^2}{N_q}\frac{A^n}{(\Delta x)^n}\,. \tag{3}$$

For Young double-slit diffraction Nesterets and Gureyev [17] used a definition of SNR corresponding to the so-called 'ideal observer SNR' [2] which quantifies the distinguishability of the image from two identical slits of width $b = A/2$ separated by distance $d = \Delta x$ from the image from one slit located in the centre and with the same width. The number of particles N_q forming each of the two images was assumed to be the same. The issue of distinguishability of such images is closely related to the Rayleigh criterion of spatial resolution [4]. It was shown that, for any fixed number N_q of image-forming quanta, the intrinsic quality, defined in equation (3), reaches its maximum at the slit separation distance d equal to $2b$, that is, when $\Delta x = A$ [17]. In other words, the number of imaging quanta required to reliably (e.g., with $\mathrm{SNR} \geqslant 5$)

distinguish an image from two identical slits from the corresponding image from one slit, reaches it minimum when $d = 2b$.

In the next two sections we investigate the relationship between inequality (2) and the Heisenberg uncertainty inequality [12]. Section 4 outlines a possible link between the noise-resolution uncertainty (2) and the notion of information capacity of communication and imaging systems, as introduced by Shannon [18]. The main results are summarized in Section 5.

2 Heisenberg and noise-resolution uncertainties

The spatial resolution of an LSI system,

$$S(\mathbf{x}) = \int T(\mathbf{x}-\mathbf{y})S_{\text{in}}(\mathbf{y})d\mathbf{y}\,, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^n\,, \tag{4}$$

is defined in terms of the width Δx of its PSF $T(\mathbf{x})$ where

$$(\Delta x)^2 = \frac{4\pi}{n}\frac{\int |\mathbf{x}|^2 T(\mathbf{x})d\mathbf{x}}{\int T(\mathbf{x})d\mathbf{x}} = \frac{4\pi\|\mathbf{x}t\|_2^2}{n\|t\|_2^2}\,, \tag{5}$$

and $T(\mathbf{x}) = |t(\mathbf{x})|^2 \geqslant 0$ is a non-negative function with finite L_1 and L_2 norms, zero first moment and finite second moment. In particular, $\int \mathbf{x}T(\mathbf{x})d\mathbf{x} = 0$, and $\|T\|_1 = \int T(\mathbf{x})d\mathbf{x} = \int |t(\mathbf{x})|^2 d\mathbf{x} = \|t\|_2^2 = \|\hat{t}\|_2^2 < \infty$, where the overhead hat symbol denotes the Fourier transform, $\hat{f}(\mathbf{u}) = \int \exp(-i2\pi\mathbf{u}\cdot\mathbf{x})f(\mathbf{x})d\mathbf{x}$.

We also define the 'angular' (or 'momentum') resolution as

$$(\Delta u)^2 = \frac{4\pi}{n\|\hat{t}\|_2^2}\int |\mathbf{u}|^2|\hat{t}(\mathbf{u})|^2 d\mathbf{u} = \frac{4\pi\|\mathbf{u}\hat{t}\|_2^2}{n\|t\|_2^2}\,. \tag{6}$$

Then the Heisenberg uncertainty inequality [12] states that

$$\Delta x\,\Delta u \geqslant 1\,. \tag{7}$$

The momentum of a mono-energetic plane-wave photon is equal to $\mathbf{p} = \hbar\mathbf{k}$, where $\mathbf{k}$ is the wave vector, $\hbar = h/(2\pi)$ and h is the Planck constant. Identifying $\mathbf{k} \equiv 2\pi\mathbf{u}$ and $\Delta p \equiv \hbar\Delta k = h\Delta u$, inequality (7) is written in a more conventional form:

$$\Delta x\, \Delta p \geqslant h\,. \tag{8}$$

The absence of the usual factor $1/(4\pi)$ on the right-hand side of the last inequality is due to the normalization factor $4\pi/n$ included in equations (5) and (6). We choose such a normalization because in the imaging context it leads to a more natural scaling of the width of PSF [13]: for example, for a rectangular PSF with the side length equal to A, from equation (5), $\Delta x = A\sqrt{\pi/3}$ in $\mathbb{R}^n$.

The noise-resolution uncertainty inequality (2) implies

$$(\Delta x)^n \geqslant C_n F_{\text{in}}^{-1} \text{SNR}^2\,. \tag{9}$$

To compare this result with the Heisenberg uncertainty principle (7), we also need an analogue of equation (9) for Δu that corresponds to equation (6).

According to the well-known properties of LSI systems [2], the SNR from equation (1) is

$$\text{SNR} = \frac{S}{\sigma} = \frac{\int S_{\text{in}}(\mathbf{y})T(\mathbf{x}-\mathbf{y})d\mathbf{y}}{\left(\int W_{\text{in}}|\hat{T}(\mathbf{u})|^2 d\mathbf{u}\right)^{1/2}} = \frac{F_{\text{in}}(\Delta x_{\text{in}})^n \|T\|_1}{W_{\text{in}}^{1/2}\|T\|_2}\,, \tag{10}$$

where $S_{\text{in}} = F_{\text{in}}(\Delta x_{\text{in}})^n$ is the (constant) input signal and W_{in} is the (constant) power spectral density of the uncorrelated noise in the input signal. Similarly to the variance of the output noise σ in the denominator of equation (10), we express the variance of the input noise via its power spectral density: $\sigma_{\text{in}}^2 = \int W_{\text{in}} d\mathbf{u} = W_{\text{in}}(\Delta x_{\text{in}})^{-n}$. As the incident fluence is spatially stationary over the entrance aperture, is spatially uncorrelated, and satisfies Poisson statistics, its variance is also constant across the aperture and is equal to $\sigma_{\text{in}}^2 = S_{\text{in}} = F_{\text{in}}(\Delta x_{\text{in}})^n$. Hence, $W_{\text{in}} = F_{\text{in}}(\Delta x_{\text{in}})^{2n}$, and substituting this expression into equation (10),

$$\mathrm{SNR}^2 = \frac{F_{in}\|T\|_1^2}{\|T\|_2^2} = \frac{F_{in}\|t\|_2^4}{\|t\|_4^4} \,. \tag{11}$$

Substituting this into equation(9), we obtain

$$(\Delta x)^n \geqslant C_n \|t\|_2^4 / \|t\|_4^4 \,, \tag{12}$$

where Δx is defined by equation (5).

A similar 'noise-resolution uncertainty' inequality is now written for Δu (as defined in (6)) by replacing $t(x)$ with $\hat{t}(u)$ in (12):

$$(\Delta u)^n \geqslant C_n \|t\|_2^4 / \|\hat{t}\|_4^4 \,. \tag{13}$$

Multiplying (12) and (13) gives us an inequality similar to the Heisenberg uncertainty (7):

$$V[t](\Delta x \, \Delta u)^n \geqslant C_n^2 \,, \tag{14}$$

where the dimensionless quantity

$$V[t] = \|t\|_4^4 \, \|\hat{t}\|_4^4 / \|t\|_2^8 \,, \tag{15}$$

represents a kind of a 'phase-space noise-to-signal ratio' (normalized with respect to the incident fluence) which characterizes a particular imaging (measuring) system.

The functional $V[t]$ is bi-invariant with respect to the scaling of its argument, that is, $V[at(b\mathbf{x})] = V[t(\mathbf{x})]$ for any positive constants a and b, hence it does not depend on the 'height' or 'width' of the function $t(\mathbf{x})$, but only on its functional form. For Gaussian functions $t_G(\mathbf{x}) = \left(a\sqrt{2\pi}\right)^{-1} \exp[-|\mathbf{x}|^2/(2b)]$, one always has $V[t_G] = 1$. In this case, inequality (14) is weaker than (7), since the Epanechnikov constants C_n are slightly smaller than 1 (for example, $C_1 = 6\sqrt{\pi/125} \cong 0.95$, $C_2 = 8/9$ and $C_3 = 60\sqrt{\pi}/7^{5/2} \cong 0.82$). Appendix A shows that the functional $V[t]$ can be arbitrarily close to zero for some functions $t(\mathbf{x})$ and can be arbitrarily large for other functions. The former means that for some functions $t(\mathbf{x})$ inequality (14) gives a stronger estimate (higher lower bound) than the Heisenberg uncertainty (7).

3 'Incoherent' version of Heisenberg uncertainty inequality

Let us define an alternative ('incoherent') angular resolution by

$$(\tilde{\Delta}u)^2 = \frac{4\pi}{n\|\hat{T}\|_1} \int |\mathbf{u}|^2 |\hat{T}(\mathbf{u})| d\mathbf{u}\,, \tag{16}$$

that is, equal to the width of the modulation transfer function $|\hat{T}(\mathbf{u})|$. We also introduce a new SNR; similarly to (11) but with $|\hat{T}(\mathbf{u})|$ in place of $T(\mathbf{x})$,

$$\widetilde{\mathrm{SNR}}^2 = \frac{F_{in}\|\hat{T}\|_1^2}{\|T\|_2^2}\,. \tag{17}$$

Then an analogue of (9) for $|\hat{T}(\mathbf{u})|$ in place of $T(\mathbf{x})$ is

$$(\tilde{\Delta}u)^n \geqslant C_n F_{in}^{-1} \widetilde{\mathrm{SNR}}^2. \tag{18}$$

Multiplying (9) and (18), we obtain:

$$(\Delta x\,\tilde{\Delta}u)^n \geqslant C_n^2 \|T\|_1^2\,\|\hat{T}\|_1^2 / \|T\|_2^4\,. \tag{19}$$

It is easy to show that $\|T\|_1^2\,\|\hat{T}\|_1^2/\|T\|_2^4 \geqslant 1$. Indeed,

$$\int T^2(x)dx = \int T(x) \int \exp(i2\pi xu)\hat{T}(u)du dx \leqslant \int |T(x)|dx \int |\hat{T}(u)|du\,.$$

Therefore,

$$\Delta x\,\tilde{\Delta}u \geqslant C_n^{2/n}. \tag{20}$$

This can be viewed as an alternative ('incoherent') form of the Heisenberg uncertainty principle.

Equation (20) is re-written as

$$4\pi^2 \frac{\int |\mathbf{x}|^2 |T(\mathbf{x})| d\mathbf{x}}{\int |T(\mathbf{x})| d\mathbf{x}} \frac{\int |\mathbf{u}|^2 |\hat{T}(\mathbf{u})| d\mathbf{u}}{\int |\hat{T}(\mathbf{u})| d\mathbf{u}} \geqslant \frac{n^2}{4} C_n^{4/n}\,. \tag{21}$$

The optimal (sharp) lower bound for the left-hand side of (21) in the 1D case is the Laue constant λ_0 [16] and Dreier et al. [8] showed that $0.543 < \lambda_0 < 0.85024$. The constant $C_1^4/4 \cong 0.205$ is much lower than the optimal bound, although strictly speaking the Laue constant is an optimal lower bound only for symmetric 1D functions $T(x)$ [16].

4 Noise-resolution uncertainty and Shannon's information capacity

Another uncertainty relationship is obtained for a (broad) class of imaging (or measuring) systems with the (output) spatial resolution not exceeding the size of the entrance aperture, that is, $\Delta x \leqslant A$. Multiplying both sides of equation (9) by $(\Delta u)^n$,

$$(\Delta x\, \Delta u)^n \geqslant C_n F_{in}^{-1} (\Delta u)^n \mathrm{SNR}^2. \tag{22}$$

As $F_{in}^{-1} = A^n / N_q$, $F_{in}^{-1}(\Delta u)^n = A^n(\Delta u)^n/N_q \geqslant 1/N_q$, because $A\Delta u \geqslant \Delta x\, \Delta u \geqslant 1$. Then

$$(\Delta x\, \Delta u)^n \geqslant C_n \mathrm{SNR}^2/N_q\,. \tag{23}$$

In one limiting case, when all output quanta are collected in a single 'detector pixel' with the Poisson statistics, we have $\mathrm{SNR}^2/N_q = 1$. In this case inequality (23) gives only a slightly smaller lower limit for its left-hand side than the conventional Heisenberg uncertainty (7). At the other limit, when the output signal is uniformly spread over multiple 'pixels' (corresponding to narrow PSFs with $A^n\|T\|_2^2 \gg \|T\|_1^2$ in (11)), the SNR can be close to 1, even for large N_q, and hence $\mathrm{SNR}^2/N_{qnt} \sim 1/N_{qnt}$, indicating that the right-hand side of (23) can in principle become arbitrarily small. Since $\mathrm{SNR}^2/N_q = Q_S^2(\Delta x)^n/A^n \leqslant Q_S^2 \leqslant 1/C_n$, hence the right-hand side of (23) is always smaller than or equal to 1, that is, it is weaker than the Heisenberg uncertainty inequality.

Inequality (23) is related to expressions for the information capacity (limits) obtained by Shannon for communication systems [18] and by Gabor and others for imaging and electromagnetic fields [11, 5]. According to Shannon [18], the number of bits N_{bits} that can be transmitted within a time interval A_t over a communication channel with bandwidth $W_t = 1/\Delta t$ is limited by

$$N_{\text{bits}} \leqslant A_t W_t \log \text{SNR}\,. \tag{24}$$

In a related result, Felgett and Linfoot [11] showed that the information capacity of a 2D (incoherent) optical system with the field of view $A_x A_y$ and the spatial bandwidth $W_x W_y = 1/(\Delta x\, \Delta y)$ is limited by

$$N_{\text{bits}} \leqslant 2 A_x W_x A_y W_y \log \text{SNR}\,. \tag{25}$$

These results were generalized further by Cox and Sheppard [5].

Returning to (23), let $W_{\text{out}} = 1/\Delta x$ be the effective output bandwidth and $A = 1/\Delta u$ be the effective output aperture of the imaging system, then (23) is re-written as

$$\text{SNR}^2 (A_{\text{out}} W_{\text{out}})^n \leqslant N_q / C_n\,. \tag{26}$$

This is quite a natural inequality as it states that:

- the maximum 'information capacity' of an imaging system is limited ultimately by the number of quanta used in the image formation;
- the size of an image, its bandwidth and the SNR (or, equivalently, the spatial and angular resolutions, and the SNR) can be traded-off against each other, but the product of the three cannot exceed the number of image-forming quanta.

Inequality (26) (noise-resolution uncertainty) in the 1D and 2D cases gives complementary results to (24) and (25). Indeed it follows from (26) that $\text{SNR}^2 A_x W_x \leqslant C_1^{-1} N_q$ and $\text{SNR}^2 A_x W_x A_y W_y \leqslant C_2^{-1} N_q$. Since $(A_{\text{out}} W_{\text{out}})^n = (A_{\text{out}}/\Delta x)^n = N_\nu$ represents the number of effective resolution units ('voxels'), the information capacity of a communication channel or an imaging system in any dimension is ultimately limited by the number of imaging (signal) quanta (e.g., photons) used:

$$N_{\text{bits}} \leqslant N_\nu \log \text{SNR}^2 \leqslant N_\nu \text{SNR}^2 \leqslant N_\nu N_q/(C_n N_\nu) \leqslant C_n^{-1} N_q \,. \qquad (27)$$

It will be interesting to consider this result in the context of scattering theory and, in particular, the limits for the information about the scatterer that can be obtained in a particular imaging scheme (e.g., a CT scan) involving a fixed number of scattering (imaging) quanta (e.g., photons). Such an investigation would be relevant to some important practical questions, for example those arising in radiation dose limited biomedical imaging or in certain astronomical problems.

5 Conclusions

We derived several forms of uncertainty inequalities which are related to the Heisenberg uncertainty principle [12] and the information capacity of communication and imaging systems described by Shannon [18] and others [11, 5]. We showed that our inequality (14) potentially provides a more accurate lower bound for the phase-space volume (which quantifies the Heisenberg uncertainty) than the conventional uncertainty relationship (7). The new lower bound is related to the phase-space noise-to-signal ratio (15) of a given imaging/measuring system. We also suggested an alternative derivation of an 'incoherent' version (20) of the Heisenberg uncertainty inequality (which may be termed the Laue inequality [8]). Finally, we obtained an estimate for the information capacity of imaging (scattering) systems which appears complementary to results of Shannon [18], Gabor and others [11, 5] concerning the information capacity of communication and imaging systems. According to this last result, the number of bits of information about the sample that can be obtained in an imaging (scattering) experiment cannot exceed the total number of the imaging quanta (probing particles) used in the experiment, while the spatial resolution (number of effective voxels) and the signal-to-noise ratio can be traded-off against each other.

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A Unboundedness of the functional $V[t]$

Recall that

$$V[t] = \frac{\|t\|_4^4 \, \|\hat{t}\|_4^4}{\|t\|_2^8} \,. \tag{28}$$

Let us assume that $V[t] \geqslant K^8$ for some $K > 0$, then, from Lemma 3.3 in Follard and Sitaram [12] it follows that this is equivalent to the inequality

$$\|t\|_4 + \|\hat{t}\|_4 \geqslant 2K\|t\|_2 \,. \tag{29}$$

However, Cowling and Price [6] showed that this inequality is not valid. Thus a non-zero lower bound does not exist.

Turning now to the upper bound, let

$$t(x) = \left(1 + x^2\right)^{-\frac{3}{8}} , \tag{30}$$

for which it is clearly the case that both $\|t\|_2$ and $\|t\|_4$ are strictly positive and finite. From Bateman and Erdelyi [3],

$$\hat{t}(\xi) = \frac{2\pi^{\frac{3}{8}} K_{\frac{1}{8}}(2\pi|\xi|)}{|\xi|^{\frac{1}{8}}\Gamma\left(\frac{3}{8}\right)}, \tag{31}$$

where K_ν is a modified Bessel function which, for small arguments, behaves as

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu} \quad \text{as } z \to 0\,. \tag{32}$$

Hence

$$\hat{t}^4(\xi) \sim \left(\frac{\pi^{\frac{1}{4}} \Gamma\left(\frac{1}{8}\right)}{\Gamma\left(\frac{3}{8}\right)} \right)^4 |\xi|^{-1} \quad \text{as } \xi \to 0\,, \tag{33}$$

and, as a consequence, $\|\hat{t}\|_4$ is unbounded. Thus $V[t]$ does not have a finite upper bound.

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