Effect of turbulence on zonal jet flows in equivalent-barotropic quasi-geostrophic model

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Abstract

The linear stability of zonal flows superposed upon a sinusoidal, slightly supercritical background flow in an equivalent-barotropic quasi-geostrophic model on a β plane is investigated using asymptotic analysis. Consistent with results of full numerical simulations, all the steady isolated zonal jet solutions were found to be linearly unstable, and specifically, the mode independent of longitudinal direction was found to be most unstable, even when the amplitude equation of the zonal flows has longitudinal dependence.

Contents

1 Introduction

It is well known that in forced two-dimensional turbulence on a rotating sphere, a multiple zonal-band structure, i.e., a structure with alternating eastward and westward jets, develops in time [1, 2]. Then, for example, in the case of the two-dimensional barotropic model, the multiple zonal-band structure experiences intermittent mergers and disappearances of zonal jets [1]. With the view to understanding such behaviour of zonal flows in two-dimensional barotropic turbulence in rotating systems, Manfroi and Young [3] considered large-scale zonal flow superposed upon a homogeneous zonal flow and a small-scale sinusoidal transverse flow on a β plane, and derived a longitude-independent amplitude equation for zonal flows. Obuse et al. [4] then obtained analytical steady isolated zonal jet solutions of Manfroi and Young's model, and investigated their linear stability. They found that the instability of the zonal jet solutions and the temporal development of the instability are consistent with the disappearance of zonal jets seen in full numerical simulations of the two-dimensional barotropic model on rotating spheres. This suggests that although Manfroi and Young's equation is a highly

simplified model of a zonal flow under the influence of zonal flow–turbulence interaction, it captures many fundamental dynamics of zonal flows in two-dimensional turbulence on a rotating sphere and a β plane. Hence, in this paper we consider the equivalent-barotropic quasi-geostrophic model, where the surface variation of the fluid layer is included in the two-dimensional barotropic system. Then we extend Manfroi and Young's model, to investigate the linear stability of zonal jets.

2   Amplitude equation and steady solution

We consider zonal flows superposed upon a small-scale sinusoidal transverse flow in an equivalent-barotropic quasi-geostrophic model, and follow the asymptotic analysis of Manfroi and Young [3], to derive an amplitude equation for zonal flows.

In the derivation of the amplitude equation of zonal flows [3, equation (4.1)], Manfroi and Young only considered a longitude-independent leading order stream function perturbation. In reality, zonal jets in the forced equivalent-barotropic quasi-geostrophic model are governed by two-dimensional equations. Therefore, we consider when the stream function is longitude-dependent.

The dynamics of incompressible flow in the forced equivalent-barotropic quasi-geostrophic model are described by the vorticity equation

$$Z_t + J(\Psi, Z) + \beta \Psi_x - \frac{1}{L_d^2} \Psi_t = F + \nu \nabla^2 Z. \tag{1}$$

Here t is the time, Ψ is the stream function, $Z = \nabla^2 \Psi$ is the vorticity, F is the vorticity forcing function, ν is the kinematic viscosity coefficient, and β is the beta parameter. We have also introduced the Rossby radius of deformation $L_d = gH/f_0^2$, where g is the magnitude of gravitational acceleration, H is the width of the zonal channel, and f_0 is the Coriolis parameter at the reference latitude, respectively. The Jacobian operator is defined as $J(A, B) = (\partial A/\partial x)(\partial B/\partial y) - (\partial A/\partial y)(\partial B/\partial x)$, and $\nabla = (\partial/\partial x, \partial/\partial y)$.

We consider the same situation as Manfroi and Young [3], i.e., a steady sinusoidal base flow with the velocity

$$(u_B, v_B) = \left(-\frac{\partial \Psi_B}{\partial y}, \frac{\partial \Psi_B}{\partial x} \right) = (U_B, m\Psi_{B0} \sin mx),$$

which is driven by a suitable forcing function, where Ψ_B, u_B and v_B are the stream function, x and y velocity components of the base flow, respectively. The parameters U_B and Ψ_{B0} are constant, and m is an integer. We also assume the Reynolds number of the base flow to be $R = \Psi_{B0}/\nu = R_c(1 + \epsilon^2)$. Here, R_c is the critical Reynolds number, and ϵ is a small quantity. Then we write the total stream function of the flow as $\Psi = \Psi_B(x, y) + \psi(x, y, t)$, where $\psi(x, y, t)$ is the perturbation stream function.

We first introduce dimensionless variables defined as

$$(\hat{x}, \hat{y}) = (mx, my), \quad \hat{t} = tm^2\nu, \quad \hat{\psi} = \frac{\psi}{\nu}, \quad \hat{u}_B = \frac{u_B}{m\nu}, \quad \hat{\beta} = \frac{\beta}{m^3\nu}, \quad (2)$$

then the stream function of the disturbance flow satisfies a non-dimensionalised equation

$$\frac{\partial \hat{\zeta}}{\partial \hat{t}} + \hat{u}_B \frac{\partial \hat{\zeta}}{\partial \hat{x}} + R\left(\frac{\partial \hat{\zeta}}{\partial \hat{y}} + \frac{\partial \hat{\psi}}{\partial \hat{y}} \right) \sin \hat{x} + J(\hat{\psi}, \hat{\zeta}) + \hat{\beta} \frac{\partial \hat{\psi}}{\partial \hat{x}} - \frac{1}{L_d^2} \frac{\partial \hat{\psi}}{\partial \hat{t}} = \hat{\nabla}^2 \hat{\zeta}, \quad (3)$$

where $\hat{\zeta} = \hat{\nabla}^2 \hat{\psi}$ is the vorticity of the disturbance flow. Hereafter, we drop the notation $\hat{\ }$ for the dimensionless variables and operators for simplicity.

Following Manfroi and Young [3], we then define[1]

$$\eta = \epsilon y, \quad \tau = \epsilon^4 t, \quad \xi = \epsilon^6 x. \quad (4)$$

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[1] The scaling of η and τ are chosen from the discussion for the band width of unstable wavenumbers and the growth rate of the instability of sinusoidal shear flow considered as a part of background flow, respectively. The scaling of ξ is chosen so that the maximum number of physical processes appear in the final amplitude equation for the perturbation [3].

Now we perform a perturbation expansion

$$\begin{cases} \psi = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots, \quad U_B = U_{B0} + \epsilon U_{B1} + \epsilon^2 U_{B2} + \cdots, \\ \beta = \beta_0 + \epsilon\beta_1 + \epsilon^2\beta_2 + \cdots, \quad \dfrac{1}{L_d} = \dfrac{1}{L_0} + \epsilon\dfrac{1}{L_1} + \cdots, \end{cases} \quad (5)$$

and multiple-scale expansion[2]

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon^6\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y} \to \epsilon\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} \to \epsilon^4\frac{\partial}{\partial \tau}. \quad (6)$$

Substituting variables and operators (4), (5), and (6) into equation (3), and seeking an amplitude equation for the $O(1)$ velocity element of the disturbance flow

$$U(\xi, \eta, \tau) = -\frac{\partial \psi_0}{\partial \eta},$$

we obtain

$$U_{\tau\eta\eta} - (1/L_1^2)U_\tau = -(2 - \gamma^2)U_{\eta\eta\eta\eta} - 3U_{\eta\eta\eta\eta\eta\eta} - 2\gamma(U^2)_{\eta\eta\eta\eta}$$
$$+ (2/3)(U^3)_{\eta\eta\eta\eta} - \beta_0 U_\xi, \quad (7)$$

where

$$\gamma = \beta_1 - U_{B1}.$$

Now, we consider a steady solution $U_0(\eta)$ of equation (7), having one extrema and $U_0 \to U_W$ as $\eta \to \pm\infty$, where U_W is a constant [4]. We first integrate (7) twice with respect to η, and use the conditions

$$U_0 \to U_W, \quad \frac{dU_0}{d\eta} \to 0 \quad \text{as } \eta \to \pm\infty.$$

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[2]Although the stream function of the disturbance flow ψ has $O(1)$ element ψ_0 in (5), the velocity of the disturbance flow is not $O(1)$, since, by considering an equation for ψ_0, it is confirmed that ψ_0 has no x−dependence [3, equations (27.a)–(28)]).

Multiplying the resulting equation by $dU_0/d\eta$, and integrating again with respect to η, we obtain

$$\frac{dU_0}{d\eta} = \pm\frac{1}{3}\sqrt{-V(U_0)}. \tag{8}$$

Here, the potential is defined as

$$V(U_0) = -U_0^4 + 4\gamma U_0^3 + 3(2-\gamma^2)U_0^2 - 6C_1 U_0 - 3C_2, \tag{9}$$

and the constants of integration C_1 and C_2 are

$$\begin{cases} C_1 = (2-\gamma^2)U_W + 2\gamma U_W^2 - \frac{2}{3}U_W^3, \\ C_2 = U_W^4 - \frac{8}{3}\gamma U_W^3 - (2-\gamma^2)U_W^2. \end{cases} \tag{10}$$

In order that the solution U_0 takes the same value U_W at $\eta \to \pm\infty$, the potential $V(U_0)$ should have a double root U_W and two other distinct real roots U_E and U_R, i.e.,

$$V(U_0) = (U_0 - U_W)^2(U_0 - U_E)(U_0 - U_R), \tag{11}$$

where $U_E < U_R$, $U_W \neq U_E$, $U_W \neq U_R$, and the conditions

$$\gamma - \frac{1}{2}\sqrt{6(\gamma^2+2)} < U_W < \gamma + \frac{1}{2}\sqrt{6(\gamma^2+2)}, \tag{12}$$

and

$$U_W < \gamma - \frac{1}{2}\sqrt{2(\gamma^2+2)} \quad \text{or} \quad \gamma + \frac{1}{2}\sqrt{2(\gamma^2+2)} < U_W, \tag{13}$$

must be satisfied.

From (9), (10), and (11), we obtain

$$U_E = 2\gamma - U_W - \sqrt{-2U_W^2 + 4\gamma U_W + \gamma^2 + 6},$$

$$U_R = 2\gamma - U_W + \sqrt{-2U_W^2 + 4\gamma U_W + \gamma^2 + 6}.$$

The steady isolated zonal jet solutions

$$U_{0\text{east}}(\eta) = \frac{a_{\text{east}}^2 U_R \tanh^2\left[\frac{(U_R - U_W)a_{\text{east}}}{6}\eta\right] - U_E}{a_{\text{east}}^2 \tanh^2\left[\frac{(U_R - U_W)a_{\text{east}}}{6}\eta\right] - 1} \, , \quad a_{\text{east}} = \sqrt{\frac{U_E - U_W}{U_R - U_W}} \, ,$$

(14)

and

$$U_{0\text{west}}(\eta) = \frac{a_{\text{west}}^2 U_E \tanh^2\left[\frac{(U_E - U_W)a_{\text{west}}}{6}\eta\right] - U_R}{a_{\text{west}}^2 \tanh^2\left[\frac{(U_E - U_W)a_{\text{west}}}{6}\eta\right] - 1} \, , \quad a_{\text{west}} = \sqrt{\frac{U_R - U_W}{U_E - U_W}} \, ,$$

(15)

for parameters γ and U_W satisfying conditions (12) and (13).

These solutions represents a zonal flow superposed upon a small-scale sinusoidal transverse background flow, governed by an amplitude equation having ξ- and η- dependence. Then $U_\xi = 0$ in equation (7) corresponds to the situation where U is ξ-independent and governed by an amplitude equation depending only on η and τ.

3   Linear stability of steady solution

To investigate the linear stability of the steady solution $U_0(\eta)$, we add a sufficiently small perturbation $v(\xi, \eta, \tau) = \exp(\sigma\tau)h(\xi)f(\eta)$. The characteristic equation is derived in the same manner as by Obuse et al. [4]. Substituting $U_0 + v$ into equation (7) and linearising with respect to v, then introducing a function $g(\eta)$, which satisfies $f(\eta) = d^2 g(\eta)/d\eta^2$ and $g \to 0$, $dg/d\eta \to 0$ as

$\eta \to \pm\infty$, we obtain the characteristic equation

$$
\sigma \left[\frac{d^2}{d\eta^2} - \frac{1}{L_1^2} \right] (gh)
$$
$$
= -(2 - \gamma^2) \frac{d^4(gh)}{d\eta^4} - 3 \frac{d^6(gh)}{d\eta^6} + \frac{d^2}{d\eta^2} \left[(2U_0^2 - 4\gamma U_0) \frac{d^2(gh)}{d\eta^2} \right] - \beta_0 g \frac{dh}{d\xi}.
$$
(16)

Since the ξ-dependence in equation (16) only appears in the form $dh(\xi)/d\xi$, by considering a Fourier expansion $h(\xi) = \sum_{n_\xi=-\infty}^{\infty} h_{n_\xi} \exp\left(in_\xi(2\pi/L_\xi)\xi\right)$, where $n_\xi \in \mathbb{N}$ is the wavenumber and L_ξ is a width of the domain in the ξ direction, it is sufficient to solve
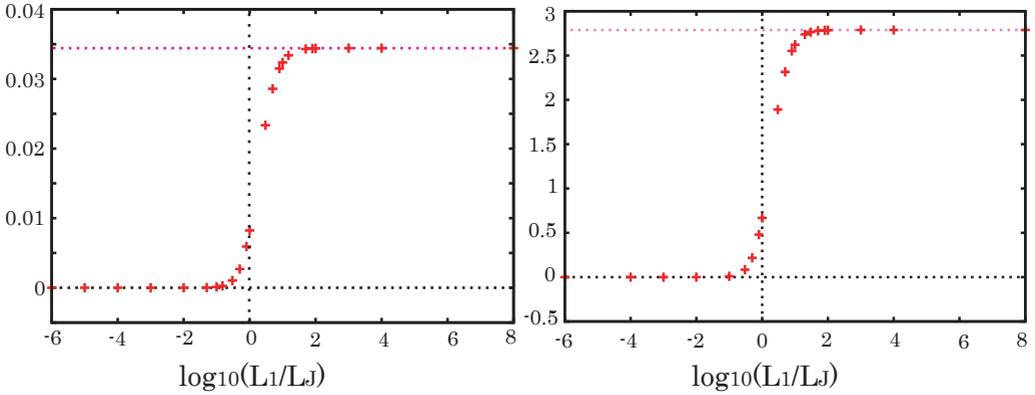
$$
\sigma \left[\frac{d^2}{d\eta^2} - \frac{1}{L_1^2} \right] g
$$
$$
= -(2 - \gamma^2) \frac{d^4 g}{d\eta^4} - 3 \frac{d^6 g}{d\eta^6} + \frac{d^2}{d\eta^2} \left[(2U_0^2 - 4\gamma U_0) \frac{d^2 g}{d\eta^2} \right] - i \frac{2\pi n_\xi}{L_\xi} \beta_0 g(\eta), \quad (17)
$$

for various $2\pi\beta_0 n_\xi/L_\xi$. Because of the symmetry property of the characteristic equation discussed by Obuse et al. [4], investigating the linear stability of U_{0east} with $\gamma \geqslant 0$ is sufficient to understand the linear stability for all U_0. It is easily verified that the characteristic equation (17) also holds for $\sigma = 0$.

We numerically solve the eigenvalue problem (17) by the Fourier spectral method, where, for example, $U_0 = \sum_{k=-K}^{K} u_k \exp\left(ik(2\pi/L_\eta)\eta\right)$. We consider a periodic domain $[0, L_\eta]$. The width of the domain, L_η, is determined so that the numerical calculations converge.[3] The size of the spatial grids is set to be $1/2^{10}$, and the truncation mode number K is chosen such that $K/L_\eta = 125/16$ for each case. The $O(\epsilon)$ element of the Rossby radius of deformation, L_1, is always expressed by the ratio L_1/L_J, where L_J is the width of the steady zonal jet U_0 and satisfies $|(U_0(0) - U_0(L_J/2))/(U_0(0) - U_W)| = 1/e$.

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[3]We mainly took $L_\eta = 384$ and 512 for each case and confirmed the convergence of the calculations.

Figure 1: Maximal growth rate of (18)(red crosses) for $(\gamma, U_W) = (1.0, -1.0)$ (left) and $(5.0, -1.0)$ (right). The pink dotted line corresponds to the equivalent result for the barotropic model.

3.1 ξ-independent case

First, consider the case where U is independent of ξ. Thus the characteristic equation is

$$\sigma \left[\frac{d^2}{d\eta^2} - \frac{1}{L_1^2} \right] g = -(2 - \gamma^2) \frac{d^4 g}{d\eta^4} - 3 \frac{d^6 g}{d\eta^6} + \frac{d^2}{d\eta^2} \left[(2U_0^2 - 4\gamma U_0) \frac{d^2 g}{d\eta^2} \right]. \quad (18)$$

The real parts of leading eigenvalues or maximal growth rate of perturbations are shown in Figure 1. When L_1/L_J is large, the maximal growth rate asymptote to those of U_0 in a barotropic system obtained by Obuse et al. [4] (the pink dotted line in Figure 1). This is to be expected, since the vorticity equation of an equivalent-barotropic quasi-geostrophic model converges to a barotropic model when $L_d \to \infty$. The maximal growth rate decreases as L_1/L_J decreases, and appear to asymptote to zero. In this limit, varying L_1/L_J only corresponds to investigating the same instability on a different time scale. Therefore the real part of the leading eigenvalue is positive, not zero, even

when $L_1/L_J = 1.0 \times 10^{-6}$. As a consequence, all the steady isolated jet solutions U_0 are linearly unstable, as for the barotropic case of Obuse et al. [4], though the instability is weaker than for the barotropic case. The instability would be expected to bring about the deformation of U_0, and cause the disappearance of zonal jets.

3.2 ξ-dependent case

Now we investigate the ξ−dependent case. Figure 2(a) shows the maximal growth rate for fixed γ, U_W and $2\pi n_\xi \beta_0/L_\xi$. For $|2\pi n_\xi \beta_0/L_\xi|$ sufficiently small, the maximal growth rate monotonically decreases as L_1/L_J decreases, similar to the ξ−independent case in Figure 1. This is to be expected, since $2\pi n_\xi \beta_0/L_\xi = 0$ corresponds to the ξ−independent case. On the other hand, for larger $|2\pi n_\xi \beta_0/L_\xi|$, the plots of the maximal growth rate show that these have maxima at $L_1/L_J \sim 10^{1/2}$. The position and magnitude of these maxima both decrease as $|2\pi n_\xi \beta_0/L_\xi|$ becomes larger, but the maximal growth rate for larger $|2\pi n_\xi \beta_0/L_\xi|$ does not exceed that for smaller $|2\pi n_\xi \beta_0/L_\xi|$ even at $L_1/L_J \sim 10^{1/2}$. Hence, the maximal growth rate for a fixed L_1/L_J monotonically decrease as $|2\pi n_\xi \beta_0/L_\xi|$ increases, and this will be confirmed later. These maxima appear for different choices of U_W and γ (not shown). The mechanism of the appearance is not yet clear, and the structure of the eigenfunctions shows no apparent change from those of large or small $|2\pi n_\xi \beta_0/L_\xi|$. Now, concentrating on the region where L_1/L_J is small, all the maximal growth rates appear to asymptote to zero. However, for the same reason as stated for the ξ−independent case in Subsection 3.1, they do not become zero but remain positive.

The plot of maximal growth rate for fixed γ, U_W and L_1/L_J is shown in Figure 2(b).[4] These monotonically decrease as $|2\pi n_\xi \beta_0/L_\xi|$ increases, and take the maximum value at $2\pi n_\xi \beta_0 = 0$, whether or not they have a lo-

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[4]We could not obtain the eigenvalue for large $|2\pi n_\xi \beta_0/L_\xi|$ because continuous modes appear and the numerical calculations no longer converge in this limit.

Figure 2: (a) Maximal growth rate for $\gamma = 1.0$, $U_W = -1.0$, $2\pi n_\xi \beta_0 / L_\xi = 1.0 \times 10^{-4}$ (red), 5.0×10^{-4} (orange), 1.0^{-3} (blue), 1.1×10^{-3} (green), and 1.2×10^{-3} (light blue). (b) Maximal growth rate (red crosses) for $\gamma = 1.0$, $U_W = -1.0$, $L_1/L_J = 1.0$(left) and 3.0(right).
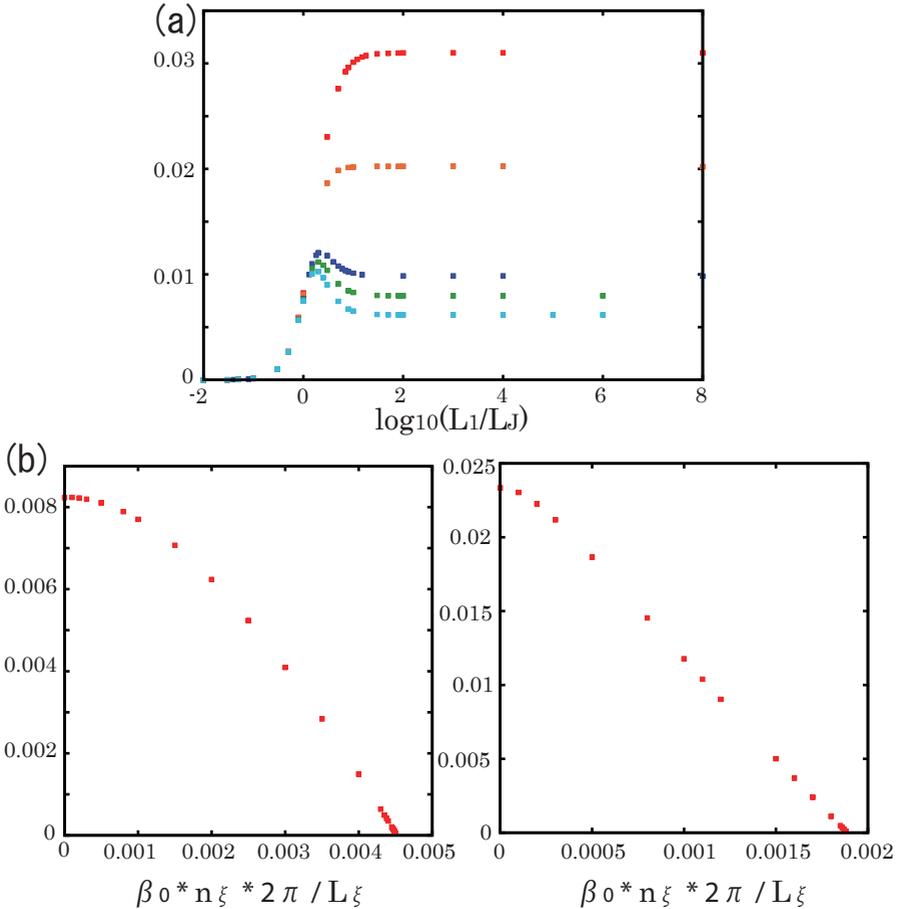
cal maximum in Figure 2(a). This confirms that the maximal growth rate at larger $|2\pi n_\xi \beta_0 / L_\xi|$ does not exceed that for smaller $|2\pi n_\xi \beta_0 / L_\xi|$ in Figure 2(a).

Furthermore, Figure 2 suggests that all the zonal jet solutions U_0 are linearly unstable, and specifically, the ξ-independent mode, i.e., the mode independent of longitudinal direction, is the most unstable mode. This is even though the amplitude equation on the zonal flows (7) has both η- and ξ- dependence. The instability is expected to bring about the deformation of the steady isolated zonal jet U_0 and cause its disappearance.

4   Conclusions

In this paper we have performed an asymptotic analysis in an equivalent-barotropic quasi-geostrophic system, extending Manfroi and Young's analysis [3]. We then studied the linear stability of steady isolated zonal jet solutions U_0 when it is governed by an amplitude equation that is independent of and dependent on the longitudinal variable ξ. It was found that in both cases all the zonal jet flows U_0 under the influence of zonal flow–nonzonal background flow interaction are linearly unstable. Consequently, the zonal jet solutions are expected to deform in time. For the case where the amplitude equation of the zonal flows has longitudinal dependence, it was also found that the mode independent of longitudinal direction, i.e., the zonal mode, is most unstable among all the two-dimensional modes. We are particularly interested in long-time behaviour and the final state of the steady isolated zonal jets U_0 in these systems, especially when U_0 is allowed to have non-zonal variation. This problem is now under investigation.

calculations were performed using the computer systems of the Institute for Information Management and Communication, Kyoto University and of the Research Institute for Mathematical Sciences, Kyoto University.

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